Identification and Optimal Reserve Prices in Ascending Auctions, with an Application to Art

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Abstract

This paper proposes a novel non-parametric approach to partial identification of the distribution of the highest valuation, seller's expected profit, and optimal reserve price in symmetric ascending auctions. Our approach restricts the distribution of valuations away from the pure common- and independent private values settings, leading to substantially tighter bounds. It also accommodates an unknown number of bidders, as long as bounds on it are available. Additionally, we formulate and solve the Min-Max-Regret problem of the seller choosing a reserve price while facing ambiguity about the distribution of valuations. We apply the proposed methodology to a new dataset consisting of more than 3500 art auctions held by the two largest auction houses, Christie's and Sotheby's. For Modern Art sold in New York City priced between \$1.0M and \$10.0M, we find that setting reserve prices 26% higher would increase the seller's expected profit by at least \$126K per lot, or \$3.8M per auction.

Keywords: Nonparametric identification, ascending auctions, partial identification, minimax regret, unknown number of bidders, correlated values, art auctions.

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1 Introduction

Open ascending auctions are widely used for selling a variety of goods ranging from used cars and timber to rare wine and art. The two major challenges in the empirical analysis of ascending auctions are unobserved heterogeneity at the auction level, which makes the valuations interdependent, and an unknown number of bidders. In this paper, we propose a new non-parametric partial identification approach, which addresses these challenges in a computationally simple manner, and apply it to art auctions.

We consider the problem of a seller choosing a reserve price to maximize expected profit under two main assumptions: (i) The bidders are symmetric, and their valuations depend on a common component and independent private components (e.g., projected future resale price of an artwork and consumption value respectively); and (ii) The transaction price in an auction is the greater of the reserve price and second-highest valuation. Under Assumption (ii), the only unknown component in the seller's expected profit is the marginal distribution of the highest valuation. Under Assumption (i), this distribution can be expressed as an expectation of a convex function of the conditional distribution of the second-highest valuation, conditional on the common component. By bounding the distance between such conditional distribution and its' marginal counterpart, we restrict the dependence between valuations away from the pure common and independent private value settings. This further allows us to obtain point-wise sharp bounds on the distribution of the highest valuation by solving two generalized moment problems that minimize or minimize the expected value of a convex function subject to constraints on the mean and variance of the underlying distribution. We show that the bounds can be easily computed through univariate convex optimization. Since the distribution of the highest valuation is not point-identified, the expected profit function is ambiguous to the seller. To resolve the ambiguity, we propose using the Min-Max-Regret criterion, arguing that it is in line with the goal of profit maximization. Since the sharp identified set for the expected profit is intractable, we formulate and analytically solve two relaxations of the Min-Max-Regret problem. Using empirically-calibrated simulations, we compare the proposed reserves with the "pessimistic" max-min reserve and the *status quo* of zero reserve and find that the proposed reserves may lead to a sizable increase in profit.

We apply the proposed methodology to study art auctions held by the two major auction houses, Christie's and Sotheby's. The auction houses serve a large market: In 2023, they reported revenues of \$7.9B and \$6.2B respectively. On November 9–10, 2022 alone, the 155-work collection of a deceased technology billionaire Paul Allen sold for \$1.65B towards philanthropy, with the most expensive piece, *Les Poseuses, Ensemble (Petite version), 1888-1890* by *Georges Seurat*, selling for \$149.24M.

Besides the challenges described above, empirical studies of art auctions are limited by data availability. The auction houses are notoriously secretive — the websites only provide basic information about the lot and the final transaction price, while the data on bids, the number of bidders, and bidders' identities in past auctions are kept private. To this end, we assemble a large new dataset comprising more than 3500 auction lots using live-stream YouTube videos of the auctions held by Sotheby's and Christie's between 2020 and 2024. For each lot, we obtain a complete bidding sequence, lower and upper bounds on the number of bidders, and detailed information about the lot, including the low and high estimates of the lot's value, location of the auction and category of art, artists name, period, provenance, and condition report. For auctions for modern art in New York City with price range between \$1.0M and \$10.0M, we find that setting reserve prices 26% higher would increase the expected seller's profit by at least \$126K per lot, or \$3.8M per auction.

Related Literature. We focus on ascending auctions and refer the reader to Athey and Haile (2007) and Hortaçsu and Perrigne (2021) for detailed discussions of the broader auction literature. Early work on non-parametric identification in ascending auctions focused primarily on the independent private values (IPV) settings with a known number of bidders. In this setting, assuming equilibrium play, Athey and Haile (2002) established point identification of the marginal distribution of bidders' valuations. Indeed, observing any order statistic of the i.i.d. sample of valuations suffices to recover the parent marginal distribution. Haile and Tamer (2003) relaxed the equilibrium play assumption, allowing for minimal bid increments and jump bidding, and provided partial identification results for the marginal distribution of valuations and expected profit. Recently, Chesher and Rosen (2017) and Molinari (2020) studied sharp identification without equilibrium play, showing that even point-wise sharp bounds on the marginal distribution of valuations are hard to characterize.

In settings with unobserved heterogeneity and interdependent valuations, the analysis is more complicated. The main challenge comes from the fact that the highest valuation is never observed, by design of the auction. Athey and Haile (2002) formally showed that in the affiliated private values model with equilibrium play, the joint distribution of bidders' valuations is not identified. Aradillas-López, Gandhi, and Quint (2013) pointed out that if the transaction price in an auction is the larger of the reserve price and the second-highest valuation, then certain functionals of interest, such as the seller's expected profit or bidder's expected surplus, depend only on the *marginal* distributions of the two highest valuations. Under the stated assumption, the distribution of the second-highest valuation is observed, so the identification problem is to obtain bounds on the distribution of the highest valuation. Assuming that the valuations are symmetric and positively dependent, the authors derived simple bounds corresponding to the extreme cases of independent private and pure common values. Observing that the bounds may be very wide, the authors proposed to leverage variation in the number of bidders to tighten the bounds, under additional restrictions. In this paper, we do not rely on the variation in the number of bidders and, instead, propose to rule out the extreme cases of independent private and pure common values, which are arguably often implausible. Our bounds are point-wise sharp and straightforward to compute.

Several approaches have been proposed to deal with unknown number of bidders. In IPV settings, a popular approach is due to Song (2004). The author noted that the distribution of an order statistic from an i.i.d. sample, conditional on a lower order statistic, does not depend on the sample size, and used this fact to non-parametrically identify the marginal distribution of the valuationss. In turn, Marra (2020) used results on stochastic ordering of the differences between adjacent order statistics, known as sample spacings, to partially identify the structural features of interest. In settings with unobserved heterogeneity, dealing with unknown number of bidders typically requires some additional information. For example, Freyberger and Larsen (2022) relied on observing the seller's profit, Mbakop (2017) and Luo and Xiao (2023) required observing more than two bids, while Hernndez, Quint, and Turansick (2020) required additional information about the distribution of the number of bidders. Instead, the approach developed in this paper restricts the "magnitude" of unobserved heterogeneity, and only requires the analyst to observe, or conservatively set, the lower and upper bounds on the number of bidders.

A few papers considered the problem of a seller selecting a reserve price in the presence of ambiguity about the distribution of valuations. In the IPV setting, Aryal and Kim (2013) proposed using the Max-Min criterion, motivating it from the Bayesian perspective. The Max-Min reserve price is simply the maximizer of the lower bound on the profit. As an alternative, Jun and Pinkse (2024) suggested maximizing the profit function corresponding to the maximum entropy distribution within the identified set, arguing that it is the least-informative choice. In this paper, we propose using the Min-Max-Regret criterion, which originated Savage (1951) and was popularized in econometrics following the work on statistical treatment rules by Manski (2004). We argue that such criterion is more in line with the goal of profit maximization than the available alternatives. Since the sharp identified set for the expected profit is intractable, we formulate and analytically solve two relaxations of the Min-Max-Regret problem. The first relaxation admits all CDF-s of the highest valuation between the available bounds. The second relaxation imposes an additional shape restriction on the distribution of valuations, under which the CDF of the highest valuation is a convex transformation of the CDF of the second-highest valuation. We solve both problems analytically and show that the proposed reserves perform well in simulations.

This paper also contributes to the literature on art and wine auctions, including Ashenfelter (1989); Ashenfelter and Graddy (2003); Beggs and Graddy (2009); Ashenfelter and Graddy (2011); McAndrew, Smith, and Thompson (2012), and Marra (2020). Although the analysis has been somewhat restricted by data availability, a common finding in the literature is that the reserve prices are too low. We further confirm this finding empirically using a large new dataset of art auctions and propose a theoretically justified way to choose a higher reserve price in practice.

The rest of the paper is organized as follows. Section 2 gives a formal setup and presents main identification results; Section 3 discusses the seller's problem under ambiguity; Section 4 briefly discusses estimation and inference; Section 5 illustrates the proposed approach with simulated data; Section 6 discusses data collection and applies the proposed methodology to art auctions; and Section 7 concludes.

2 Identification in Ascending Auctions

2.1 Environment and Bidding Behavior

Consider an open ascending auction with N bidders with symmetric private values and a secret reserve price. Let $V_1, \ldots, V_N \in \mathcal{V}$ denote the valuations, $V_{1:N}, \ldots, V_{N:N}$ denote the corresponding order statistics, and $F_{j:N}$ denote the distribution of $V_{j:N}$, conditional on N, for $j \in \{1, \ldots, N\}$. Let $v_0 \in \mathcal{V}$ denote the value of unsold good to the seller, and $r \in \mathbb{R}_+$ the reserve price. Following Aradillas-López, Gandhi, and Quint (2013), we impose the following assumption on the bidding behavior.

Assumption 2.1 (Transaction Price)**.** *The transaction price is the greater of the reserve price r and the second-highest valuation, $V_{N-1:N}$.*

In the standard button auction model with equilibrium play, Assumption 2.1 holds exactly. In many empirical settings, it holds approximately if bidders do not "jump bid" at the end of the auction. We find this assumption plausible in our dataset and discuss possible relaxations in Section 7. Under Assumption 2.1, the seller's realized profit is

$$(r - v_0) \underbrace{\mathbf{1}(V_{N:N} \geqslant r, V_{N-1:N} < r)}_{\text{only one active bidder}} + (V_{N-1:N} - v_0) \underbrace{\mathbf{1}(V_{N-1:N} \geqslant r)}_{\text{at least two bidders}}.$$

Taking expectation, conditional on N,

$$\pi_N(r) = \int_0^{+\infty} \max(r, v) dF_{N-1:N}(v) - v_0 - (r - v_0) F_{N:N}(r). \tag{1}$$

Importantly, the expected profit only depends on the marginal distributions of the second-highest and highest valuations. Since the auction ends before the highest bidder fully reveals their valuation, the main identification problem is to obtain useful bounds on $F_{N:N}(\cdot)$. We start with the following general setting.

Assumption 2.2 (Conditional Independence)**.** *(i) In an auction with N bidders, their valuations take the form $V_j = g(U, \varepsilon_j)$, for some i.i.d. random variables $\varepsilon_1, \dots, \varepsilon_N \in \mathbb{R}$, random vector $U \in \mathbb{R}^{d_U}$, independent of the ε_j-s, conditional on N, and a measurable function $g : \mathbb{R}^{d_U} \times \mathbb{R} \to \mathcal{V}$. (ii) The map $e \mapsto g(u, e)$ is weakly increasing, for all u.*

Condition (i) states that valuations are symmetric and consist of a common component U and independent private components ε_j, so that $V_1, \dots, V_N$ are i.i.d., conditional on U. In the context of art auctions, U may represent, for example, projected future resale value, and ε_j — individual consumption value. We do not take a stance on whether the bidders observe U or not. Condition (i) implies positive dependence between valuations:[1] for any measurable function $h : \mathcal{V} \to \mathbb{R}$ and a non-empty subset of bidders $J \subseteq \{1, \dots, N\}$,

$$\mathbb{E}\left[\prod_{j \in J} h(V_j) \,\middle|\, N\right] = \mathbb{E}\left[\mathbb{E}\left[h(V_j) \,|\, U, N\right]^{|J|} \,\middle|\, N\right] \geqslant \mathbb{E}[\, h(V_j) \,|\, N]^{|J|},$$

using the Law of Iterated Expectations and Jensen's inequality. In particular, the above implies $\mathbb{C}ov(h(V_i), h(V_j) \,|\, N) \geqslant 0$, for all $i \neq j$, for all $h(\cdot)$. Importantly, Condition (i) restricts the sign of the dependence between the valuations but not its degree, allowing for the two extreme cases of pure common and independent private values. In turn, Condition (ii) will be used for tractability in the sequel.

Assumption 2.2 (i) allows to derive simple bounds on $F_{N:N}(\cdot)$ in terms of the observed $F_{N-1:N}(\cdot)$. Consider a function $\phi_N : [0,1] \to [0,1]$ defined implicitly via the relation:

$$u = N\phi_N(u)^{N-1} - (N-1)\phi_N(u)^N. \tag{2}$$

This function maps the CDF of the second-highest value out of N i.i.d. draws to the corresponding marginal CDF. The function $u \mapsto \phi_N(u)$ is strictly increasing and smooth on $(0,1)$, and, as we show in Lemma A.1, the function $u \mapsto \phi_N(u)^N$ is convex, for all $N \geqslant 1$.

Proposition 1 (Bounds on $F_{N:N}$ Under Conditional Independence)**.** *Let Assumption 2.2*

[1] By the de Finetti-Hewitt-Savage Theorem, Condition (i) is equivalent to $V_1, \dots, V_N$ being part of an infinitely exchangeable sequence. It is more restrictive than finite exchangeability precisely because of the positive dependence, as explained below.

(i) hold and the function ϕ_N be defined in (2). Then, for all $v \in \mathcal{V}$,

$$\phi_N(F_{N-1:N}(v))^N \leqslant F_{N:N}(v) \leqslant F_{N-1:N}(v).$$

The equality $F_{N:N}(v) = \phi_N(F_{N-1:N}(v))^N$, for all v, corresponds to independent private values, and the equality $F_{N:N}(v) = F_{N-1:N}(v)$, for all v, corresponds to pure common value.

Aradillas-López, Gandhi, and Quint (2013) obtained the same bounds under a weaker notion of positive dependence.[2] We derive the bounds directly from conditional independence and take them as a starting point of the analysis. To elaborate, note that under Assumption 2.2 (i), by the Law of Iterated Expectations:

$$F_{N:N}(v) = \mathbb{E}[\phi_N(P(V_{N-1:N} \leqslant v \,|\, U, N))^N \,|\, N]$$
$$F_{N-1:N}(v) = \mathbb{E}[P(V_{N-1:N} \leqslant v \,|\, U, N) \,|\, N].$$

Denoting $\mu = F_{N-1:N}(v)$ and $X = P(V_{N-1:N} \leqslant v \,|\, U, N)$, for any fixed $v \in \mathcal{V}$, the bounds in Proposition 1 correspond to the solutions of the following generalized moment problems:

$$\begin{aligned} &\min_P\{\mathbb{E}_P[\phi_N(X)^N] : \mathbb{E}_P[X] = \mu, P(X \in [0,1]) = 1\} = \phi_N(\mu)^N; \\ &\max_P\{\mathbb{E}_P[\phi_N(X)^N] : \mathbb{E}_P[X] = \mu, P(X \in [0,1]) = 1\} = \mu. \end{aligned} \tag{3}$$

The lower bound corresponds to Jensen's inequality and the upper bound to the so-called Edmundson-Madansky's inequality for the convex function $u \mapsto \phi_N(u)^N$ on $u \in [0,1]$ (Edmundson, 1957; Madansky, 1959). In the absence of additional restrictions on the distribution of $P(V_{N-1:N} \leqslant v \,|\, U)$, these bounds are sharp.

Since Assumption 2.2 (i) does not restrict the degree of dependence between valuations, the bounds in Proposition 1 are typically very wide. However, the assumption of pure common or independent private values may not be realistic in many empirical settings, and it is desirable to control thedependence between valuations more precisely. To this end, we additionally restrict the variance of $P(V_{N-1:N} \leqslant v \,|\, U, N)$, conditional on N, denoted:

$$D_N(v) = \mathbb{E}[(P(V_{N-1:N} \leqslant v \,|\, U, N) - F_{N-1:N}(v))^2 \,|\, N]. \tag{4}$$

The variance corresponds to the squared $L_2(P)$ distance between the conditional and unconditional CDFs of $V_{N-1:N}$, thus providing a measure of the "magnitude" of the common

[2] The authors assume that the valuations are exchangeable and the function $f(k) = P(V_i \leqslant v \,|\, \#\{j \neq i : V_j \leqslant v\} = k)$ is non-decreasing in k. Lemma 1 in the aforementioned paper shows that conditional independence implies the stated property.

component in valuations. Since $P(V_{N-1:N} \leqslant v \mid U, N)$ is a random variable supported within $[0,1]$, the variance $D_N(v)$ takes values in $[0, F_{N-1:N}(v)(1 - F_{N-1:N}(v))]$. The marginal cases, $D_N(v) = 0$ and $D_N(v) = F_{N-1:N}(v)(1 - F_{N-1:N}(v))$, correspond to independent private values and pure common value correspondingly.[3] Thus, bounding the variance function $D_N(v)$ away from the extremes allows to control the degree of dependence between valuations.

To obtain an interpretable expression for $D_N(v)$, we invoke Assumption 2.2 (ii). Under this assumption, the second-highest valuation is $V_{N-1:N} = g(U, \varepsilon_{N-1:N})$, so letting $f_{U|N}$ denote the density of U, conditional on N, we have

$$\begin{aligned}\mathbb{E}[P(V_{N-1:N} \leqslant v \mid U, N)^2 \mid N] &= \int P(g(u, \varepsilon_{N-1:N}) \leqslant v \mid N)^2 f_{U|N}(u) du \\ &= P(g(U, \varepsilon_{N-1:N}) \leqslant v, g(U, \tilde{\varepsilon}_{N-1:N}) \leqslant v \mid N) \\ &= C_N(F_{N-1:N}(v), F_{N-1:N}(v)),\end{aligned}$$

where $\tilde{\varepsilon}_{N-1:N}$ is an independent copy of $\varepsilon_{N-1:N}$, and $C_N : [0,1]^2 \to [0,1]$ is the copula function of the vector $(V_{N-1:N}, \tilde{V}_{N-1:N}) = (g(U, \varepsilon_{N-1:N}), g(U, \tilde{\varepsilon}_{N-1:N}))$, conditional on N.[4] The copula function captures the dependence between the second highest valuation in two identical auctions with the same common component and two independent draws of the private components.[5] Letting $C_0 : [0,1]^2 \mapsto [0,1]$ denote the independence copula, defined as $C_0(u, v) = uv$, the variance $D_N(v)$ can be written as:

$$D_N(v) = C_N(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v)).$$

This expression motivates the following assumption.

Assumption 2.3 (Departure from IPV and Pure Common Values)**.** *Let* $C_\rho : [0,1]^2 \to [0,1]$ *be a copula function, parametrized by* $\rho \in [0,1]$ *such that: (i)* $C_0(u_1, u_2) = u_1 u_2$, $C_1(u_1, u_2) = \min(u_1, u_2)$*; (ii) for each* u, $\rho \mapsto C_\rho(u, u)$ *is non-decreasing; (iii) for each* ρ,

[3] Notice $D_N(v) = 0$, for all v, implies that $P(V_{N-1:N} \leqslant v \mid U, N) = F_{N-1:N}(v)$ almost surely, which, under Assumption 2.2 (i), implies that either U is constant or $V_j = g(\varepsilon_j)$, corresponding to the i.i.d. valuations. On the other hand, $D_N(v) = F_{N-1:N}(v)(1 - F_{N-1:N}(v))$, for all v, can only be attained if $P(V_{N-1:N} \leqslant v \mid U, N) \in \{0, 1\}$ such that $P(\{P(V_{N-1:N} \leqslant v \mid U, N) = 1\} \mid N) = F_{N-1:N}(v)$, which can only happen if $V_{N-1:N}$ is a deterministic function of U, corresponding to the pure common values.

[4] The copula function of random variables X and Y with a joint CDF F_{XY} is defined as a function $C : [0,1]^2 \to [0,1]$ satisfying $F_{XY}(x, y) = C(F_X(x), F_Y(y))$. If X and Y are continuous, the copula represents the joint distribution of the random variables $F_X(X)$ and $F_Y(Y)$, which are marginally Uniform $[0,1]$.

[5] If the researcher observes two repeated auctions of identical goods with independent draws of N bidders, and the transaction price reveals second highest valuation in each of the auctions, the copula function is identified from the data. Pursuing this formally is beyond the scope of the paper.

$(C_\rho(u,u))' \geqslant \max\{2\frac{u-C_\rho(u,u)}{1-u}, \frac{C_\rho(u)}{u}\}$. *The variance* $D_N(v)$*, defined in* (4)*, satisfies:*

$$\underline{D}_N(v) \leqslant D_N(v) \leqslant \overline{D}_N(v)$$

with

$$\underline{D}_N(v) = C_{\underline{\rho}}(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v))$$

$$\overline{D}_N(v) = C_{\overline{\rho}}(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v)),$$

for some $0 \leqslant \underline{\rho} \leqslant \overline{\rho} \leqslant 1$.

Intuitively, ρ may be interpreted as the share of the common component in valuations, and $\underline{\rho}, \overline{\rho}$ are lower and upper bounds on this share. Choosing $\underline{\rho} > 0$ ensures a non-trivial common component, and $\overline{\rho} < 1$ ensures a non-trivial private component. Setting $\underline{\rho} = \overline{\rho}$ fixes $D_N(v)$ but leaves the joint distribution of $(V_{N-1:N}, \tilde{V}_{N-1:N})$ outside of the diagonal (v, v) completely unspecified. In such case, without further assumptions, the CDF of the highest valuation $F_{N:N}(v)$ is not point identified. We highlight that Assumption 2.3 does not impose a specific dependence structure on the valuations; it simply provides a coherent way to choose the bounding functions $\underline{D}_N(v)$ and $\overline{D}_N(v)$ for the analysis. Our results do not rely on the specific parametric form of $C_\rho(u,u)$ in any way.

Condition (i) of the Assumption maintains independent private values and pure common value as the marginal cases; Condition (ii) ensures that the assumed bounds are ordered properly; Condition (iii) is a shape restriction that guarantees that the bounds derived in Theorem 1 below are plausible CDFs. In practice, it suffices to choose a function $C_\rho(u,u)$ that interpolates between u and u^2 depending on the value of $\rho \in [0,1]$. Lemma A.2 in the Appendix verifies that natural choices, such as $C_\rho(u,u) = \rho u + (1-\rho)u^2$, $C_\rho(u,u) = u^{2-\rho}$, or $C_\rho(u,u)$ corresponding to the bivariate Gaussian copula with correlation $\rho \in [0,1]$, satisfy all of the stated assumptions. Any other bivariate copula satisfying the stated assumptions and re-parametrized appropriately can be used (see, e.g., Table 4.1. in Nelsen, 2006, for the examples of Archimedian copulas).

2.2 Bounding CDF of The highest Valuation and Expected Profit

Under Assumptions 2.2 and 2.3, pointwise sharp bounds on $F_{N:N}$ can be obtained by solving generalized moment problems similar to the ones in (3) with an extra constraint of the form $c_1 \leqslant \mathbb{V}ar(X) \leqslant c_2$. Using geometric arguments of Kemperman (1968), we show that both bounds can be computed by solving univariate convex optimization problems.

Theorem 1 (CDF Bounds Conditional on N)**.** *Let Assumptions 2.2 and 2.3 hold. Then,* *for each* $v \in \mathcal{V}$*,*

$$\underline{\psi}_N(F_{N-1:N}(v)) \leqslant F_{N:N}(v) \leqslant \overline{\psi}_N(F_{N-1:N}(v)),$$

where:

$$\begin{aligned}
\underline{\psi}_N(F_{N-1:N}(v)) &= \min_{s\in[\frac{c_1}{1-\mu},\mu]} \left\{\phi_N(\mu-s)^N \tfrac{c_1}{c_1+s^2} + \phi_N\left(\mu+\tfrac{c_1}{s}\right)^N \tfrac{s^2}{c_1+s^2}\right\};\\
\overline{\psi}_N(F_{N-1:N}(v)) &= \max_{s\in\left[\mu-\frac{c_2}{1-\mu},\mu+\frac{c_2}{\mu}\right]} \left\{\mu - \tfrac{\mu(1-\mu)-c_2}{s(1-s)}(s-\phi_N(s)^N)\right\},
\end{aligned}$$

with $\mu = F_{N-1:N}(v)$*,* $c_1 = \underline{D}_N(v)$*, and* $c_2 = \overline{D}_N(v)$*. The minimization problem is strictly convex, and the maximization problem is strictly concave. Moreover, the bounding functions* $\underline{\psi}_N(\cdot)$ *and* $\overline{\psi}_N(\cdot)$ *are monotonically increasing and smooth in their argument.*

For $\underline{\rho} = 0$ and $\overline{\rho} = 1$, the bounds coincide with those in Proposition 1, because they solve the moment problems in (3). For any $0 < \overline{\rho} \leqslant \underline{\rho} < 1$, the bounds in Theorem 1 are strictly tighter. If the only information available to the econometrician is $F_{N-1:N}$ (i.e., only the transaction price is observed), the bounds are pointwise sharp. Since the bounds are monotonically increasing, they are plausibly sharp uniformly over $v \in \mathcal{V}$, although it is hard to show formally. Additionally, we note that certain CDFs within the bounds do not belong to the sharp identified set because they cannot correspond to the CDF of order statistics of conditional i.i.d. valuations. For example, if $F_{N-1:N}(v)$ does not have atoms ("jumps"), $F_{N:N}(v)$ cannot have atoms either, so all such CDFs will be excluded form the identified set. Although the sharp identified set is very hard to describe, the proposed bounds are sufficiently informative, as we show in the sequel.

Theorem 1 implies the following bounds on the expected profit.

Theorem 2 (Bounds on Expected Profit Conditional on N)**.** *Suppose Assumptions 2.1, 2.2, and 2.3 hold. Let* $\underline{\psi}_N(F_{N-1:N}(v))$ *and* $\overline{\psi}_N(F_{N-1:N}(v))$ *denote the lower and upper bounds on* $F_{N:N}(v)$ *in Theorem 1. Then, the expected profit, conditional on* N*, is bounded by:*

$$\begin{aligned}
\pi_N(r) &\geqslant \int_0^\infty \max\{r,v\} dF_{N-1:N}(v) - v_0 - \overline{\psi}_N(F_{N-1:N}(r))(r-v_0);\\
\pi_N(r) &\leqslant \int_0^\infty \max\{r,v\} dF_{N-1:N}(v) - v_0 - \underline{\psi}_N(F_{N-1:N}(r))(r-v_0),
\end{aligned}$$

for all $r \geqslant v_0$*.*

As in Theorem 1, the bounds are sharp point-wise in r and plausibly sharp uniformly. Likewise, many profit functions within the bounds cannot be attained under the stated assumptions. The bounds on expected profit for r below v_0 can be obtained by switching the roles of $\underline{\psi}_N(\cdot)$ and $\overline{\psi}_N(\cdot)$, but they are irrelevant because it is never optimal to set the reserve price below v_0.[6]

2.3 Unknown Number of Bidders

In ascending auctions, the exact number of bidders may often be unknown to the econometrician and potentially even to the seller. In this section, we show that the above analysis can be easily extended to the case when only bounds on the number of bidders are available.

Assumption 2.4 (Unknown Number of Bidders)**.** *The number of bidders N is unknown but satisfies $\underline{N} \leqslant N \leqslant \overline{N}$ for some known $2 \leqslant \underline{N} \leqslant \overline{N}$.*

In such context, a natural objective function is the unconditional expected profit. Taking expectations over N in (1) yields:

$$\mathbb{E}[\pi_N(r)] = \int_0^\infty \max\{r,v\} dF_2(v) - v_0 - F_1(r)(r - v_0), \tag{5}$$

where

$$F_2(v) = \mathbb{E}[F_{N-1:N}(v)] = P(V_{N-1:N} \leqslant v);$$
$$F_1(v) = \mathbb{E}[F_{N:N}(v)] = P(V_{N:N} \leqslant v)$$

denote the marginal distributions of $V_{N-1:N}$ and $V_{N:N}$, correspondingly. Although N is not observed, under Assumption 2.1, the marginal distribution $F_2(v)$ is identified by the data, so the identification problem is to bound $F_1(\cdot)$. To this end, we apply the same ideas as in the preceding subsection.

By Lemma A.1 in Appendix, for each $t \in [0,1]$, the map $N \mapsto \phi_N(t)^N$, with $\phi_N(\cdot)$ defined in (2), is increasing. Using this fact and the Law of Iterated Expectations, the marginal CDF of $V_{N:N}$ is bounded by

$$\mathbb{E}[\phi_{\underline{N}}(P(V_{N-1:N} \leqslant v \,|\, U, N))^{\underline{N}}] \;\leqslant\; F_1(v) \;\leqslant\; \mathbb{E}[\phi_{\overline{N}}(P(V_{N-1:N} \leqslant v \,|\, U, N))^{\overline{N}}]. \tag{6}$$

and, additionally, we know that

$$F_2(v) = \mathbb{E}[P(V_{N-1:N} \leqslant v \,|\, U, N)].$$

Let

$$D(v) = \mathbb{E}[(P(V_{N-1:N} \leqslant v \,|\, U, N) - F_2(v))^2] \tag{7}$$

[6] This follows directly from the first-order condition for maximizing $\pi_N(r)$ in (1).

be the unconditional analog of $D_N(v)$ defined in (4). In order to bound $D(v)$ away from independent private and pure common values, we modify Assumptions 2.2 and 2.3 as follows.

Assumption 2.2$^+$ (Conditional Independence)**.** *In an auction with N bidders, (i) valuations take the form $V_j = g(U, \varepsilon_j)$, for some i.i.d. random variables $\varepsilon_1, \ldots, \varepsilon_N \in \mathbb{R}$, random vector $U \in \mathbb{R}^{d_U}$, and a measurable function $g : \mathbb{R}^{d_U} \times \mathbb{R} \to \mathcal{V}$. (ii) The map $e \mapsto g(u, e)$ is weakly increasing, for all u. (iii) (U, N) is jointly independent of $(\varepsilon_1, \ldots, \varepsilon_N)$.*

Notice that Condition (iii) does not restrict the dependence between the common component and the number of bidders, but requires full independence of the private components. In the context of art auctions, this assumption allows for a situation that artworks by more popular artists attract more bidders, but does not allow those bidders to have a different distribution of tastes. Under Assumption 2.2$^+$, repeating the arguments in the preceding section unconditionally, the variance $D(v)$ can be expressed as:

$$D(v) = C(F_2(v), F_2(v)) - C_0(F_2(v), F_2(v)),$$

where $C : [0,1]^2 \to [0,1]$ denotes the copula of $(V_{N-1:N}, \tilde{V}_{N-1:N}) = (g(U, \varepsilon_{N-1:N}), g(U, \tilde{\varepsilon}_{N-1:N}))$, and C_0 the independence copula.

Assumption 2.3$^+$ (Departure from IPV and Pure Common Values)**.** *Let $C_\rho : [0,1]^2 \to [0,1]$ be a copula function, parametrized by $\rho \in [0,1]$ such that: (i) $C_0(u_1, u_2) = u_1 u_2$, $C_1(u_1, u_2) = \min(u_1, u_2)$; (ii) for each u, $\rho \mapsto C_\rho(u, u)$ is non-decreasing; (iii) for each ρ, $(C_\rho(u,u))' \geqslant \max\{2\frac{u - C_\rho(u,u)}{1-u}, \frac{C_\rho(u)}{u}\}$. The variance $D(v)$, defined in (7), satisfies:*

$$\underline{D}(v) \leqslant D(v) \leqslant \overline{D}(v)$$

with

$$\underline{D}(v) = C_{\underline{\rho}}(F_2(v), F_2(v)) - C_0(F_2(v), F_2(v))$$

$$\overline{D}(v) = C_{\overline{\rho}}(F_2(v), F_2(v)) - C_0(F_2(v), F_2(v)),$$

for some $0 \leqslant \underline{\rho} \leqslant \overline{\rho} \leqslant 1$.

Note that since $C(F_2(v), F_2(v)) = \mathbb{E}[C_N(F_{N-1:N}(v), F_{N-1:N}(v))]$, by the Law of Iterated Expectations, Assumption 2.3 implies Assumption 2.3$^+$. The interpretation of $\underline{\rho}, \overline{\rho}$ and the arguments leading to the choice of bounding functions remain the same. The following results are direct analogs of Theorems 1 and 2.

Theorem 3 (CDF Bounds With Unknown N)**.** *Let Assumptions 2.2$^+$ and 2.3$^+$ hold. Then, for each $v \in \mathcal{V}$,*

$$\underline{\psi}_{\underline{N}}(F_2(v)) \leqslant F_1(v) \leqslant \overline{\psi}_{\overline{N}}(F_2(v)),$$

where:

$$\begin{aligned}\underline{\psi}_{\underline{N}}(F_2(v)) &= \min_{s\in[\frac{c_1}{1-\mu},\mu]} \left\{\phi_{\underline{N}}(\mu-s)^{\underline{N}}\tfrac{c_1}{c_1+s^2} + \phi_{\underline{N}}\left(\mu+\tfrac{c_1}{s}\right)^{\underline{N}}\tfrac{s^2}{c_1+s^2}\right\};\\ \overline{\psi}_{\overline{N}}(F_2(v)) &= \max_{s\in\left[\mu-\frac{c_2}{1-\mu},\mu+\frac{c_2}{\mu}\right]} \left\{\mu - \tfrac{\mu(1-\mu)-c_2}{s(1-s)}\left(s-\phi_{\overline{N}}(s)^{\overline{N}}\right)\right\},\end{aligned}$$

with $\mu = F_2(v)$*,* $c_1 = \underline{D}(v)$*, and* $c_2 = \overline{D}(v)$*. The minimization problem is strictly convex, and the maximization problem is strictly concave. Moreover, the bounding functions* $\underline{\psi}_N(\cdot)$ *and* $\overline{\psi}_N(\cdot)$ *are monotonically increasing and smooth in their argument.*

Theorem 4 (Bounds on Expected Profit With Unknown N)**.** *Let Assumptions 2.1, 2.2*$^+$*, 2.3*$^+$*, and 2.4 hold. Let* $F_1(\cdot)$ *and* $F_2(\cdot)$ *denote the marginal distributions of* $V_{N:N}$ *and* $V_{N-1:N}$ *correspondingly. Then,* $F_2(\cdot)$ *is identified by the data, and, for each* r*:*

$$\begin{aligned}\mathbb{E}[\pi_N(r)] &\geqslant \int_0^\infty \max\{r,v\}dF_2(v) - v_0 - \overline{\psi}_{\overline{N}}(F_2(r))(r-v_0);\\ \mathbb{E}[\pi_N(r)] &\leqslant \int_0^\infty \max\{r,v\}dF_2(v) - v_0 - \underline{\psi}_{\underline{N}}(F_2(r))(r-v_0),\end{aligned}$$

where $\underline{\psi}_{\underline{N}}(\cdot)$ *and* $\overline{\psi}_{\overline{N}}(\cdot)$ *are the bounding functions defined in Theorem 3.*

As in Theorems 1 and 2, the bounds are point-wise sharp. The lower bound can be binding only if $N = \overline{N}$, and the upper bound — only if $N = \underline{N}$, almost surely.

3 Optimal Reserve Prices

3.1 Resolving Ambiguity

Consider the problem of the seller setting a reserve price to maximize expected profit from an auction. Under our assumptions on information structure and bidding behavior, the expected profit function is only partially identified, so the target is ambiguous. In this section, we discuss existing approaches to resolving the ambiguity and advocate for using the so-called Min-Max-Regret approach.

Let $\pi(\cdot)$ denote the true expected profit function, Π_0 — the set of all plausible profit functions, and $\mathcal{R}_0$ — the corresponding set of optimal reserve prices. There are three prevalent approaches to resolving ambiguity in the literature: Bayesian, Max-Min, and Min-Max-Regret; see, e.g., Manski (2022). The Bayesian approach is to assume a prior distribution Q over Π_0 and solve:

$$\sup_{r\in\mathcal{R}_0} \int_{\Pi_0} \pi(r)dQ(\pi).$$

In our setting, Π_0 represents the sharp identified set for $\pi(\cdot)$, so any prior within the identified set cannot be updated based on the data. On the other hand, the prior directly affects the proposed solution, and by using different priors (e.g., point masses on the elements of Π) one can recover any point $r \in \mathcal{R}_0$ as the "optimal" solution. Thus, the Bayesian approach directly imposes further restrictions on the model and is conceptually not helpful in the present setting. The Max-Min (MM) approach is to solve:

$$\sup_{r \in \mathcal{R}_0} \inf_{\pi \in \Pi_0} \pi(r),$$

i.e., maximize the lower bound on the expected profit function. Intuitively, this corresponds to setting the reserve price cautiously, which may not align with the goals of the auction house. Indeed, if the lot is unsold, the marginal cost associated with organizing its resale is likely negligible, compared to the selling price.[7] Nevertheless, as we show in Section 5, the max-min solution may perform well.

The Min-Max-Regret (MMR) approach is to solve:

$$\inf_{r \in \mathcal{R}_0} \sup_{\pi \in \Pi_0} \{\pi(r_\pi^*) - \pi(r)\},$$

where r_π^* denotes an optimal reserve price under the profit function π. To gain intuition, let $\phi^* : \Pi_0 \to \mathbb{R}$, given by $\phi^*(\pi) = \pi(r_\pi^*)$, return the maximum of π, and for each $r \in \mathcal{R}_0$, let $\phi_r : \Pi_0 \to \mathbb{R}$, given by $\phi_r(\pi) = \pi(r)$, evaluate π at a given reserve price r. Then, the MMR problem can be equivalently stated as:

$$\inf_{r \in \mathcal{R}_0} \sup_{\pi \in \Pi_0} |\phi^*(\pi) - \phi_r(\pi)|. \tag{8}$$

That is, choosing r is equivalent to choosing a functional $\phi_r(\cdot)$ as close as possible to the profit-maximizing functional $\phi^*(\cdot)$ uniformly over unknown $\pi \in \Pi_0$. Conceptually, this is in line with the goal of profit maximization under ambiguity.[8]

The MMR problem is formulated with the sharp identified sets Π_0 and $\mathcal{R}_0$. When these are intractable, one may consider a "relaxed" problem with some outer sets Π and $\mathcal{R}$:

$$\inf_{r \in \mathcal{R}} \sup_{\pi \in \Pi} |\phi^*(\pi) - \phi_r(\pi)|. \tag{9}$$

The interpretation remains the same: one is choosing a functional $\phi_r(\cdot)$ as close as possible to

[7] In our dataset, the auction houses may sell nearly fifty lots in one auction.

[8] In some settings, the MM and MMR solutions may coincide. We find that it is typically not the case in our setting, and both solutions may perform well in practice; see Section 5.

the profit-maximizing functional $\phi^*(\cdot)$ uniformly over unknown $\pi \in \Pi$. Intuitively, if $\Pi \backslash \Pi_0$ is not too large, and $\phi^*(\cdot)$ is well-behaved on $\Pi \backslash \Pi_0$, the solutions of (8) and (9) should be close to each other. Therefore, the relaxed MMR criterion in (9) provides a reasonable way of choosing the reserved price under ambiguity.

3.2 The Min-Max-Regret Problem

Theorem 1 derived lower and upper bounds on $F_{N:N}$ under Assumptions 2.2 and 2.3. While the bounds are point-wise sharp, certain CDFs between the bounds may not be admissible under the conditional independence assumption. For example, if $F_{N-1:N}$ does not have flat regions or jumps, then $F_{N:N}$ cannot have those either. Unfortunately, the sharp identified set for $F_{N:N}$ and thus for the expected profit function is hard to characterize, which makes the MMR problem in (8) intractable. In this section, we consider two relaxations of the MMR problem that can be solved analytically. Our results take bounds on $F_{N:N}$ as given, and hold for any bounds satisfying the stated assumptions going beyond the setting of Theorem 1. The discussion is presented conditional on N, but the results apply similarly to the unconditional case with obvious modifications of the assumptions.

3.2.1 CDFs Between the Bounds

First, we consider the following setting.

Assumption 3.1 (Relaxed MMR). *(i) valuations are exchangeable; (ii) The transaction price is the greater of the reserve price and the second-highest valuaion; (iii) The CDF of the highest value satisfies $\underline{F}(v) \leqslant F_{N:N}(v) \leqslant \overline{F}(v)$ for some known weakly increasing functions $\underline{F}(v)$, $\overline{F}(v)$ such that $0 \leqslant \underline{F}(v) \leqslant \overline{F}(v) \leqslant F_{N-1:N}(v)$, for all $v \in \mathcal{V}$.*

Condition (i) assumes that valuations are symmetric but does not impose conditional i.i.d. structure as in Assumption 2.2; Condition (ii) is a re-statement of Assumption 2.1; and Condition (iii) states that any CDF $F_{N:N}$ between the available bounds is admissible (e.g. for the bounds in Theorem 1). To ensure coherency with Condition (i), Lemma A.5 in the Appendix shows that any pair $(F_{N-1:N}(v), F_{N:N}(v))$ with $F_{N:N}(v)$ between the bounds can be the CDF's of the top two order statistics of exchangeable random variables.

Assumption 3.1 defines a set of expected profit functions for the relaxed MMR problem. Plugging $\overline{F}(r)$ and $\underline{F}(r)$ instead of $F_{N:N}(r)$ in Equation (1) yields the respective lower and upper bounds on the profit, but not all functions between the bounds are admissible profit functions. For example, expected profit cannot arbitrarily "jump" up because $F_{N:N}(r)$ must be non-decreasing. Nevertheless, the relaxed MMR solution is easy to characterize.

Theorem 5 (Relaxed MMR Solution). *Let Assumption 3.1 hold. Denote the expected profit in* (1) *by* $\pi_N(r; F_{N-1:N}, F_{N:N})$. *Then, for any* $r \geqslant v_0$, *the max-regret is given by* $R(r) = \max\{R_1(r), R_2(r)\}$, *where:*

$$R_1(r) = \max_{v_0 \leqslant v \leqslant r} \pi_N(v; F_{N-1:N}, F^*_{1,r}) - \pi_N(r; F_{N-1:N}, F^*_{1,r})$$

$$R_2(r) = \max_{v \geqslant r} \pi_N(v; F_{N-1:N}, F^*_{2,r}) - \pi_N(r; F_{N-1:N}, F^*_{2,r})$$

with $F^*_{1,r}(v)$ *and* $F^*_{2,r}(v)$ *given by:*

$$F^*_{1,r}(v) = \mathbf{1}(v < r)\underline{F}(v) + \mathbf{1}(v \geqslant r)\overline{F}(v) \tag{10}$$

$$F^*_{2,r}(v) = \mathbf{1}(v < r)\overline{F}(v) + \mathbf{1}\{r \leqslant v < \bar{v}(r)\}\overline{F}(r) + \mathbf{1}\{v \geqslant \bar{v}(r)\}\underline{F}(v) \tag{11}$$

where $\bar{v}(r) = \min\{v : \underline{F}(v) \geqslant \overline{F}(r)\}$. *The Min-Max-Regret reserve is* $r^* = \operatorname{argmin}_{r \geqslant v_0} R(r)$.

The proof is straightforward. Given any reserve r, the max-regret is attained either below or above it. We show that the max-regret realizes below r for a profit function that "jumps down" at r, corresponding to the CDF $F^*_{1,r}$, and above r for a profit function that increases "as much as possible" after r, corresponding to the CDF $F^*_{2,r}$. Solving for the MMR optimal reserve r^* numerically is straightforward. In fact, since $R_1(r)$ is increasing and $R_2(r)$ is decreasing (at least locally around the point of their intersection), r^* solves $\Delta(r) = R_1(r) - R_2(r) = 0$, where $\Delta(r)$ is smooth and locally increasing.

3.2.2 CDFs with a Shape Restriction

Next we consider an additional restriction that induces more structure on the problem.

Assumption 3.2 (Convex MMR). *(i) Valuations are exchangeable; (ii) The transaction price is the greater of the reserve price and the second-highest valuation; (iii) The CDF of the highest valuation satisfies* $\underline{\psi}(F_{N-1:N}) \leqslant F_{N:N}(v) \leqslant \overline{\psi}(F_{N-1:N})$ *for some increasing, twice differentiable, convex functions* $\underline{\psi}, \overline{\psi} : [0,1] \to [0,1]$; *(iv) Letting* $f_{V_{N:N}}(v)$ *and* $f_{V_{N-1:N}}(v)$ *denote the density functions of the highest and second-highest valuations, the likelihood ratio* $f_{V_{N:N}}(v)/f_{V_{N-1:N}}(v)$ *is increasing in* v.

Conditions (i)–(ii) are the same as before. Condition (iii) imposes a shape restriction on the bounds on $F_{N:N}$, which is satisfied, e.g., by the bounds in Theorems 1 and 3. Condition (iv), known as the likelihood ratio dominance, is a strong form of stochastic dominance. Intuitively, it means that if an observer sees a realization $V = v$ of a random variable drawn either from $f_{V_{N:N}}$ or from $f_{V_{N-1:N}}$ but does not know with certainty which one, then the higher $V = v$ is, the more likely it was drawn from $f_{V_{N:N}}$ rather than $f_{V_{N-1:N}}$. Importantly, Condition (iv) by itself does not restrict the strength of dependence between valuations, and it is easy to verify that it holds under IPV and pure common values. It also holds for many commonly used copulas, such as Gaussian, Clayton, or Frank copulas.[9]

As shown in Lemma A.6 in the Appendix, exchangeability and likelihood ratio dominance imply that

$$F_{N:N}(v) = g_N(F_{N-1:N}(v)),$$

for some convex function $g_N : [0, 1] \to [0, 1]$, for all $v \in \mathcal{V}$. This fact substantially refines the set of admissible CDFs between the bounds and allows to solve the corresponding MMR problem analytically, as we do below.

Theorem 6 (Convex MMR Solution)**.** *Let Assumption 3.2 hold. Denote the expected profit in* (1) *by* $\pi_N(r; F_{N-1:N}, F_{N:N})$*. Then, for any* $r \geqslant v_0$*, the max-regret is given by* $R(r) = \max\{R_1(r), R_2(r)\}$*, where:*

$$R_1(r) = \max_{v_0 \leqslant v \leqslant r} \pi_N(v; F_{N-1:N}, F^*_{1,r}) - \pi_N(r; F_{N-1:N}, F^*_{1,r})$$

$$R_2(r) = \max_{v \geqslant r} \pi_N(v; F_{N-1:N}, F^*_{2,r}) - \pi_N(r; F_{N-1:N}, F^*_{2,r})$$

where $F^*_{j,r}(v) = g_{j,r}(F_{N-1:N}(v))$ *for convex functions* $g_j : [0, 1] \to [0, 1]$ *defined as follows. Denoting* $u_r = F_{N-1:N}(r)$*,*

$$g_{1,r}(u) = \begin{cases} \underline{\psi}(u) & u \leqslant \underline{u}_{1,r} \\ a_{1,r}u + b_{1,r} & \underline{u}_{1,r} < u \leqslant u_r \\ \overline{\psi}(u) & u > u_r, \end{cases}$$

where $\underline{u}_{1,r}$ *solves* $\overline{\psi}(u_r) = \overline{\psi}'(u_r)(u_r - \underline{u}_{1,r}) + \underline{\psi}(\underline{u}_{1,r})$*,* $a_{1,r} = \overline{\psi}'(u_r)$*, and* $b_{1,r} = \overline{\psi}(u_r) - a_{1,r}u_r$*. Further:*

$$g_{2,r}(u) = \begin{cases} \overline{\psi}(u) & u \leqslant u_r \\ a_{2,r}u + b_{2,r} & u_r < u \leqslant \overline{u}_{2,r} \\ \underline{\psi}(u) & u > \overline{u}_{2,r}, \end{cases}$$

where $\bar{u}_{2,r}$ *solves* $\underline{\psi}(\bar{u}_{2,r}) - \overline{\psi}'(u_r)(\bar{u}_{2,r} - u_r) = \overline{\psi}(u_r)$*,* $a_{2,r} = \overline{\psi}'(u_r)$*, and* $b_{2,r} = \overline{\psi}(u_r) - a_{2,r}u_r$*. The Min-Max-Regret reserve is* $r^* = \operatorname{argmin}_{r \geqslant v_0} R(r)$*.*

[9] See e.g. Nelsen (2006) or Joe (2014) for reviews of the copula theory. The likelihood ratio dominance is equivalent to $[V_{N:N} \,|\, a \leqslant V_{N:N} \leqslant b] \geqslant_{\text{st}} [V_{N-1:N} \,|\, a \leqslant V_{N-1:N} \leqslant b]$, for any $a, b \in \mathcal{V}$ with $a < b$, where $\geqslant_{\text{st}}$ denotes first-order stochastic dominance; see, e.g., Chapter 1.C. in Shaked and Shanthikumar (2007).

The proof idea is similar to that of Theorem 5, except the "worst-case" CDFs $F^*_{1,r}$ and $F^*_{2,r}$ are now required to be convex transformations of $F_{N-1:N}$. The transformation $g_{1,r}$ corresponds to the CDF that decreases "as fast as possible" just below r, and $g_{2,r}$ to the CDF that increases "as fast as possible" above r, while preserving convexity. As in Theorem 5, solving for r^* numerically is straightforward.

4 Estimation and Inference

For brevity, the exposition below treats the number of bidders N as known and fixed, but the extension to varying and/or unknown N is immediate. We assume that the econometrician has access to a sample of n independent auctions, $(P_i, X_i)_{i=1}^n$, where $X_i \in \mathcal{X}$ is vector of auction-specific covariates, and $P_i \in \mathbb{R}_+$ is the transaction price in auction i, which is assumed equal to the second-highest valuation $V_{N-1:N,i}$.

4.1 Bounds on Expected Profit

The profit bounds in Theorem 2 are known, deterministic functions of the observable distribution $F_{N-1:N}(v) = P(V_{N-1:N} \leqslant v \,|\, N)$. The best approach to estimating this distribution depends on the context. In addition to the number of bidders N, one may want to condition on a vector of covariates X. If the dataset is small, or X is very rich, non-parametric estimation may be imprecise. In such cases, one may impose a flexible parametric model for $P(V_{N-1:N} \leqslant v \,|\, N, X)$, similar to Athey, Levin, and Seira (2011), or assume that $P(V_{N-1:N} \leqslant v \,|\, N, X)$ depends on X only through an index $X'\gamma$ and employ, e.g., the semi-parametric MLE of Ai (1997). When non-parametric estimation is suitable, the bandwidths may be selected using a reference parametric model; see Ichimura and Todd (2007).

Letting $\underline{\pi}(r; F_{N-1:N})$ and $\overline{\pi}(r, F_{N-1:N})$ denote the profit bounds in Theorem 2 and $\hat{F}_{N-1:N}$ be a suitable estimator for $F_{N-1:N}$, the profit bounds are estimated as:

$$\begin{aligned} \hat{\underline{\pi}}(r) &= \underline{\pi}(r; \hat{F}_{N-1:N}); \\ \hat{\overline{\pi}}(r) &= \overline{\pi}(r; \hat{F}_{N-1:N}). \end{aligned} \tag{12}$$

Since the bounding functions $\underline{\psi}_N$ and $\overline{\psi}_N$ are convex, the plug-in estimators for the bounds are generally biased downward. The bias may be corrected using standard jackknife or bootstrap techniques; see, e.g., Wasserman (2006). In turn, point-wise and uniform confidence bands can be constructed using delta-method.

In our empirical application, there are no continuous covariates, so we use the sample analog estimator $\hat{F}_{N-1:N}$. Additionally, we find that the finite-sample bias of $\underline{\psi}_N(\hat{F}_{N-1:N})$ and $\overline{\psi}_N(\hat{F}_{N-1:N})$ is negligible given the sample size and thus do not use any bias-correction methods. Appendix B spells out the asymptotic distributions of $\hat{\underline{\pi}}(r)$ and $\hat{\overline{\pi}}(r)$ in this case.

4.2 Bounds on Expected Profit under MMR Reserve Price

First, consider estimating the MMR optimal reserve price r^* obtained in Theorems 5 or 6. Letting $R_j(r; F_{N-1:N})$ for $j \in \{1, 2\}$ denote the corresponding maximum regrets, the MMR optimal reserve price r^* solves:[10]

$$\Delta(r; F_{N-1:N}) \equiv R_1(r; F_{N-1:N}) - R_2(r; F_{N-1:N}) = 0.$$

Denoting $\hat{\Delta}(r) = \Delta(r; \hat{F}_{N-1:N})$, a natural estimator $\hat{r}$ for r^* solves:

$$\hat{\Delta}(\hat{r}) = 0. \tag{13}$$

Under the standard regularity conditions, consistency of $\hat{r}$ follows from the uniform consistency of $\hat{F}_{N-1:N}$ and continuity of $F \mapsto \Delta(\cdot; F)$ with respect to the sup-norms. In Appendix B, using the mean-value expansion of (13) around r^* and continuity of $F \mapsto \Delta'(\cdot; F)$ with respect to the sup-norms, we further establish that

$$\sqrt{n}(\hat{r} - r^*) \quad \to_d \quad N(0, V_r). \tag{14}$$

Next, consider the expected profit under MMR reserve price, $\pi^* = \pi_N(r^*)$. It is bounded by $\pi^* \in [\underline{\pi}^*, \overline{\pi}^*] \equiv [\underline{\pi}_N(r^*), \overline{\pi}_N(r^*)]$. The natural estimators for $\hat{\underline{\pi}}^*$ and $\hat{\overline{\pi}}^*$ are:

$$\begin{aligned} \hat{\underline{\pi}}^* &= \hat{\underline{\pi}}(\hat{r}); \\ \hat{\overline{\pi}}^* &= \hat{\overline{\pi}}(\hat{r}), \end{aligned} \tag{15}$$

where $\hat{\underline{\pi}}$ and $\hat{\overline{\pi}}$ are defined in (12), and $\hat{r}$ in (13). Consistency of these estimators follows immediately from consistency of $\hat{r}$ and continuity of the maps $F \mapsto \underline{\pi}(\cdot; F)$ and $F \mapsto \overline{\pi}(\cdot; F)$ with respect to the sup-norms. For the asymptotic distribution, consider an expansion:

$$\begin{aligned} \sqrt{n}(\hat{\underline{\pi}}^* - \underline{\pi}^*) &= \{\sqrt{n}(\hat{\underline{\pi}}(\hat{r}) - \underline{\pi}(\hat{r})) - \sqrt{n}(\hat{\underline{\pi}}(r^*) - \underline{\pi}(r^*))\} & (I) \\ &+ \sqrt{n}(\hat{\underline{\pi}}(r^*) - \underline{\pi}(r^*)) & (II) \\ &+ \sqrt{n}(\underline{\pi}(\hat{r}) - \underline{\pi}(r^*)). & (III) \end{aligned} \tag{16}$$

[10]That is, the minimizer of $R(r) = \max(R_1(r), R_2(r))$ is given by the intersection of $R_1(r)$ and $R_2(r)$.

In Appendix B, using stochastic equicontinuity arguments, we show that term (I) converges in probability to zero. The term $(II) + (III)$ is easily seen to be asymptotically Normal. Applying similar arguments to the upper bound estimator $\hat{\overline{\pi}}^*$, it follows that:

$$\begin{aligned} \sqrt{n}(\hat{\underline{\pi}}^* - \underline{\pi}^*) &\quad \to_d \quad N(0, V_{\underline{\pi}^*}); \\ \sqrt{n}(\hat{\overline{\pi}}^* - \overline{\pi}^*) &\quad \to_d \quad N(0, V_{\overline{\pi}^*}). \end{aligned} \tag{17}$$

Then, given consistent estimators $\hat{V}_{\underline{\pi}}$ and $\hat{V}_{\overline{\pi}}$, a $(1-\alpha)$ confidence interval for $\pi^* \in [\underline{\pi}^*, \overline{\pi}^*]$ can be constructed as:

$$\left[\hat{\underline{\pi}}^* - z_{1-\alpha}\sqrt{\frac{\hat{V}_{\underline{\pi}^*}}{n}}, \;\; \hat{\overline{\pi}}^* + z_{1-\alpha}\sqrt{\frac{\hat{V}_{\overline{\pi}^*}}{n}} \right]. \tag{18}$$

Such confidence interval is uniformly asymptotically valid over the set of distributions bounded away from point identification; see Imbens and Manski (2004), Stoye (2009), and Canay and Shaikh (2017) for related discussions. In our setup, point identification is ruled out by construction of the bounds, so the above confidence interval is appropriate.

The asymptotic variances in (17) are very cumbersome due to the dependence between the terms (II) and (III) in (16). However, the above discussion also implies that the distributions of $\sqrt{n}(\hat{\underline{\pi}}^* - \underline{\pi}^*)$ and $\sqrt{n}(\hat{\overline{\pi}}^* - \overline{\pi}^*)$ are continuous in the underlying distribution of the data, so a valid confidence interval can also be constructed using bootstrap as follows.

Algorithm 1 (Bootstrap CI)**.**

1. *Using the original sample, compute* $\hat{\underline{\pi}}^*$ *and* $\hat{\overline{\pi}}^*$ *according to* (15)*.*

2. *Draw a large number of bootstrap samples* $(P^*_{i,b}, X^*_{i,b})^n_{i=1}$*, for* $b = 1, \ldots, B$*, and compute* $\hat{\underline{\pi}}^*_b$ *and* $\hat{\overline{\pi}}^*_b$ *according to* (15)*, for each* b*.*

3. *Letting* $\hat{\underline{q}}_{1-\alpha}$ *denote the* $(1-\alpha)$ *empirical quantile of* $\{\sqrt{n}(\hat{\underline{\pi}}^*_b - \hat{\underline{\pi}}^*)\}^B_{b=1}$*, and* $\hat{\overline{q}}_\alpha$ *denote the* α *empirical quantile of* $\{\sqrt{n}(\hat{\overline{\pi}}^*_b - \hat{\overline{\pi}}^*)\}^B_{b=1}$*, compute the confidence interval:*

$$\left[\hat{\underline{\pi}}^* - \frac{\hat{\underline{q}}_{1-\alpha}}{\sqrt{n}}, \;\; \hat{\overline{\pi}}^* - \frac{\hat{\overline{q}}_\alpha}{\sqrt{n}} \right]. \tag{19}$$

We remark that although Step 2 involves solving the MMR problem for each bootstrap sample, the procedure is computationally simple.

5 Monte Carlo Experiments

In this section, we illustrate performance of the proposed approach in a controlled environment. First, we show that the CDF bounds in Theorems 1 and 3 and the implied profit bounds in Theorems 2 and 4 are substantially tighter than those previously available in the literature. Second, we show that choosing the reserve price using Min-Max-Regret criteria in Theorems 5 and 6, combined with the new tighter identified sets, may lead to significant increase in seller's profit relative to the status quo.

5.1 Simulation Design

The simulation design is calibrated to four stylized facts in our empirical application.

1. *The auction houses report the low and the high estimates of each lot's value. These estimates are crude and not necessarily backed by any market or historical research.* Normalizing the high estimate to 1, the low estimate is around 2/3, on average. We consider the value of unsold good to the seller, v_0, to be either equal to the low estimate, $v_0 = 2/3$, or the mid point between low and high estimates, $v_0 = 5/6$.

2. *The reserve price is secret and does not exceed the lot's low estimate.* In simulations, we set the reserve price to zero.

3. *There is significant variation in the number of active bidders per auction.* We draw the number of bidders randomly for each auction from the set $N \in \{2, 3, \dots, 10\}$, and use $\underline{N} = 2$ and $\overline{N} = 10$ throughout the simulations.

4. *According to the bidding guidelines, the minimal bid increment is 10% of the standing bid, but this rule is not strictly enforced.* We simulate auctions as follows. Given the initial set of N bidders, the first bid by a randomly drawn bidder is cast at 5% of their valuation. At each subsequent iteration, an active bidder is drawn at random from the pool of remaining bidders, whose valuations are above the standing bid. With probability $1 - \lambda$, the new bid is equal to 1.1 of the standing bid, and with probability λ, the new bid is drawn uniformly between the standing bid and 1.1 of the standing bid. If the new bid is above the active bidder's valuation, the bidder is eliminated, and a new bidder is drawn from the pool of remaining bidders. The iterations proceed until only one bidder remains.

5.2 Simulation Results

5.2.1 Distribution of The Highest Valuation and Expected Profit

First, we illustrate the CDF bounds from Theorems 3 and 4. The valuations are generated as $\log(V_j) = U + \varepsilon_j$, where $U \sim N(\mu_U, \sigma_U^2)$ and the distribution of ε_j is such that $\varepsilon_{N-1:N} \sim N(\mu_\varepsilon, \sigma_\varepsilon)$. The parameters $\mu_U, \mu_\varepsilon, \sigma_U$, and σ_ε are chosen to match the average transaction price in the data, $\mathbb{E}[V_{N-1:N}] = 1.35$. The copula function used to define the bounds in Assumption 2.3 matches the above data-generating process and corresponds to $\rho = 0.8$.

Figure 1 presents bounds on the CDF of $V_{N:N}$ (unconditionally), and the corresponding bounds on expected profit. The black solid line represents the true CDF, and the black dashed lines — the IPV and pure common value bounds. The blue dashed lines represent lower and upper bounds with $\underline{\rho} = \overline{\rho} = 0.8$, and the green lines — with $\underline{\rho} = 0.7$ and $\overline{\rho} = 0.9$. The two main takeaways are: (i) the proposed bounds are substantially tighter then the IPV and common value bounds even for a relatively wide range of $\rho \in [\underline{\rho}, \overline{\rho}] = [0.7, 0.9]$; and (ii) even when correctly specifying $\underline{\rho} = \rho = \overline{\rho} = 0.8$, we are still far from point identification. We find very similar results conditional on N.

5.2.2 Optimal Reserve Prices

Next, we compare the MMR reserves of Theorems 5 and 6 with the MM reserves and *status quo* of $r = v_0$. We consider $\underline{\rho} = 0, \overline{\rho} = 1$ and $\underline{\rho} = \overline{\rho} = 0.8$, and $v_0 = 5/6$.

Figure 2 presents the results. The left panel depicts the MM reserves, which maximize the corresponding profit lower bounds. With $\underline{\rho} = 0, \overline{\rho} = 1$, the lower bound is monotonically decreasing, so the MM reserve is v_0. With $\underline{\rho} = \overline{\rho} = 0.8$, the MM reserve is 1.1, which appears reasonable and is much closer to the true optimum of 1.33. The right panel depicts the Max-Regret functions from Theorems 5 and 6 and corresponding MMR reserves. With $\underline{\rho} = 0, \overline{\rho} = 1$, we find that both Relaxed MMR and Convex MMR reserves coincide with v_0. In contrast, with $\underline{\rho} = \overline{\rho} = 0.8$, the relaxed MMR reserve appears close to MM (1.11 vs. 1.10), while the convex MMR reserve is nearly at the optimum (1.31 vs. 1.33). Table 1 reports percentage gains in expected profit and shows additionally that the Convex MMR solution with the common component bounds set at $\underline{\rho} = 0.7$ and $\overline{\rho} = 0.9$ is able to realize nearly all available profit gains.

Thus, we find that the MM, relaxed MMR, and especially convex MMR reserves computed with the new tight bounds on the CDF of the highest valuation are close to the optimum and result in a sizable increase in expected profit.

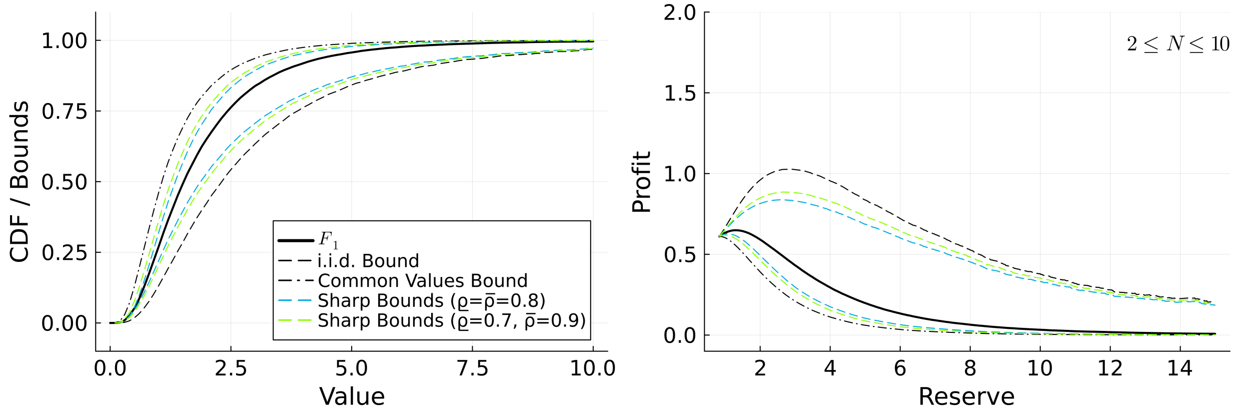


Figure 1: Simulated bounds on the CDF of the highest valuation and expected profit.

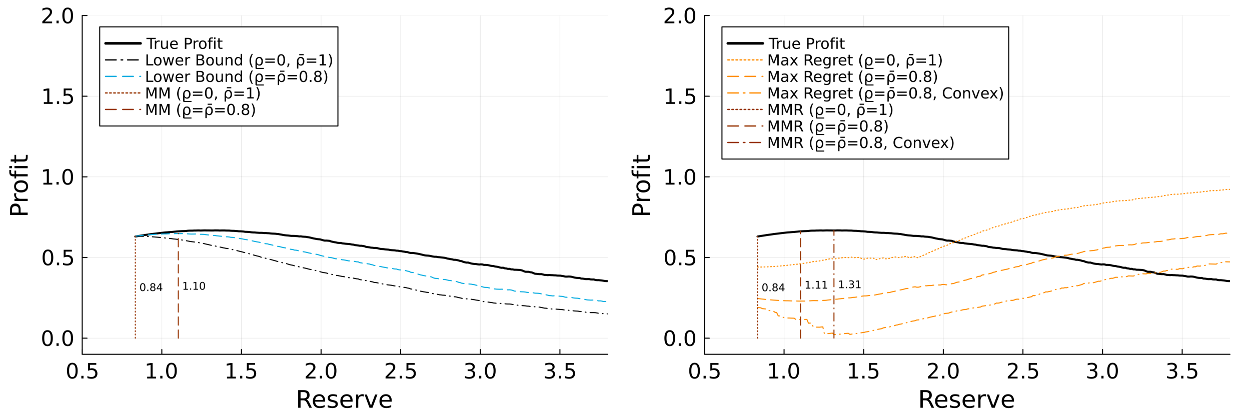


Figure 2: Simulated profit bounds and optimal reserve prices.

	Reserve	**Profit**	**Increase over $r = v_0$**
True Optimal	1.33	0.668	6.03%
MM ($\underline{\rho} = 0.8, \overline{\rho} = 0.8$)	1.10	0.662	5.01%
MM/Both MMR ($\underline{\rho} = 0.0, \overline{\rho} = 1.0$)	0.84	0.632	0.00%
Relaxed MMR ($\underline{\rho} = 0.8, \overline{\rho} = 0.8$)	1.11	0.663	5.24%
Convex MMR ($\underline{\rho} = 0.7, \overline{\rho} = 0.9$)	1.15	0.665	5.56%
Convex MMR ($\underline{\rho} = 0.8, \overline{\rho} = 0.8$)	1.31	0.668	6.03%

Table 1: Simulated profit under suggested reserve prices.

Category	Subcategory	Location	Count
Art	Chinese Art	New York	112
Art	Impressionist/20th/21st Century Art	Hong Kong	636
Art	Impressionist/20th/21st Century Art	Las Vegas	9
Art	Impressionist/20th/21st Century Art	London	708
Art	Impressionist/20th/21st Century Art	New York	1208
Art	Impressionist/20th/21st Century Art	Shanghai	55
Art	Old Masters	London	102
Art	Old Masters	New York	134
Wine/jewelry/etc.	–	–	542
Total			3506

Table 2: The art auction dataset.

6 Application: Reserve Prices in Art Auctions

6.1 Data

The auctions are held in an open ascending format, and, on average, thirty lots are sold in each auction. Prior to the auction, besides the basic information about each lot, the auction houses publish a low and high estimates of the lot's value. According to the auction houses, these estimates may not be backed by any market or historical research, and are intended as a crude reference for prospective bidders. The reserve price is kept secret, but is known to be less than the lower estimate of the lot's value. The auctioneer may bid on behalf of the seller to ensure the reserve is met.

6.1.1 Bids and Lot Information

For each auction lot, we obtain a complete bidding trajectory by applying computer vision techniques to frame-by-frame video data. Figure 3 presents examples of the frames. We crop the bottom sections, highlighted in red, and use an optical character recognition package Tesseract[11] to extract the lot's information and the bidding sequence. Then, we match the bids for each lot with the lot's characteristics scraped from the auction houses' websites. We use the Ratcliff-Obershelp strings matching algorithm[12] to accommodate any errors or misspellings in Tesseract's output. Table 2 summarizes the auctions across art categories and locations. Due to the nature of the data, we cannot match the bids with the bidders. Thus, we only use the two highest bids, which arguably belong to different bidders.

[11] Available at https://github.com/tesseract-ocr/tesseract.

[12] Available in a Python package `difflib` command `SequenceMatcher`.

6.1.2 Number of Bidders

The exact number of bidders in each auction cannot be reliably identified from the auction videos, so we obtain bounds on it using the audio transcripts.[13] In each auction, there are four types of bidders in each auction: (i) telephone, (ii) live, (iii) absentee, and (iv) online; see Figure 4. We obtain a lower bound on the true number of bidders by counting the number of unique bidders or each type.

Each telephone bidder is represented by an employee of the auction house, whom the auctioneer calls by their names. For example, in the top-right panel of Figure 4, the Christie's auctioneer references Olivier Camu, who is putting a telephone bid on behalf of an unknown buyer. To identify the names from the transcript, we use the so-called Named Entity Recognition language models. To minimize errors, we apply two state-of-the-art language models, RoBERTa (Liu, Ott, Goyal, Du, Joshi, Chen, Levy, Lewis, Zettlemoyer, and Stoyanov, 2019) and Contextual String Embeddings (Akbik, Blythe, and Vollgraf, 2018), and combine their output matching similar strings and removing duplicates. To capture the identities of the unique live, absentee, and online bidders, we look of keywords {"online bidder" = "online," "sir" = "gentleman," "madam" = "lady," "back of the room," "to the right", "to the left," "absentee"}. Our approach allows to capture all of the telephone bidders, up to one unique online or absentee bidder, and up to five unique live bidders. The distribution of the lower bound on the number of bidders is shown in Figure 5. Although it is unclear how tight the resulting lower bound is, the existing statistics are reassuring: in 2021, 42% of winning bids in Christie's live auctions were made over the telephone and only 7% in the salesroom,[14] meaning that at least some of the "missing" bidders may not have affected the auction's outcome anyway.

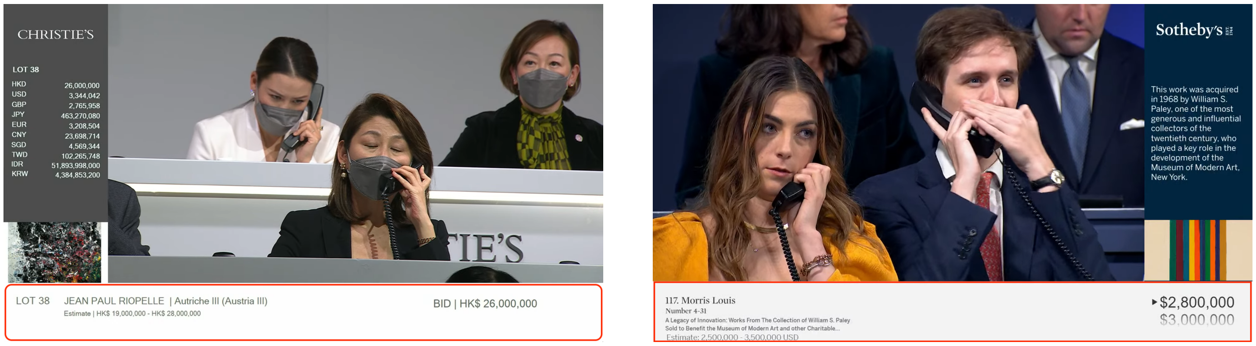
We conservatively set the upper bound on the number of bidders to two times the lower bound. An even more conservative alternative would be the total number of bids. However, we find that the function $\phi_N(t)^N$, which is used to construct the bounds, converges pointwise as N increases, so for $N, N' \geqslant 10$ the difference between $\phi_N(t)^N$ and $\phi_{N'}(t)^{N'}$ becomes negligible. Thus, the choice between conservative upper bounds does not affect the results.

6.1.3 Auction House Fees

The auction houses charge around 25% buyer's premium and 10% seller's commission with some variation depending on location and lot type. Table 8 in the Appendix summarizes the Buyer's Premium Schedule, for both Christie's and Sotheby's, as of March 2023. We adjust

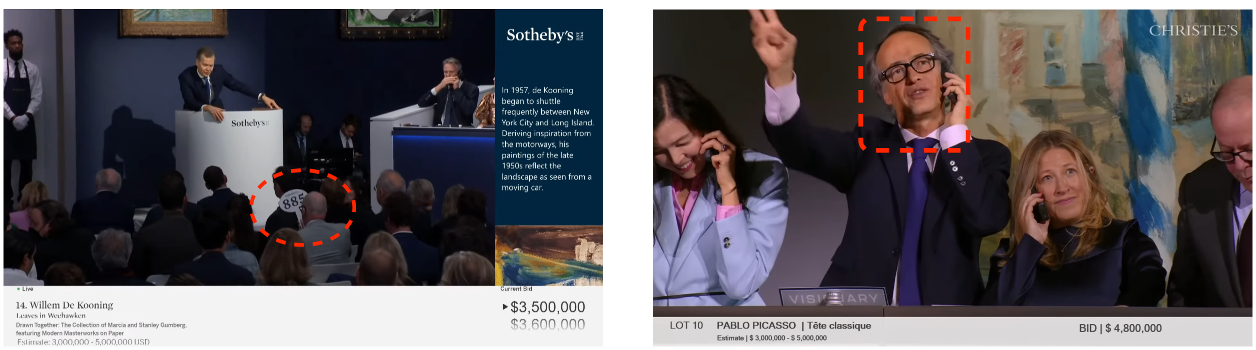
[13] The transcripts are generated using Google Cloud Platform Speech-to-Text Recognition model. See https://cloud.google.com/speech-to-text for the details.

[14] See the following article: link.

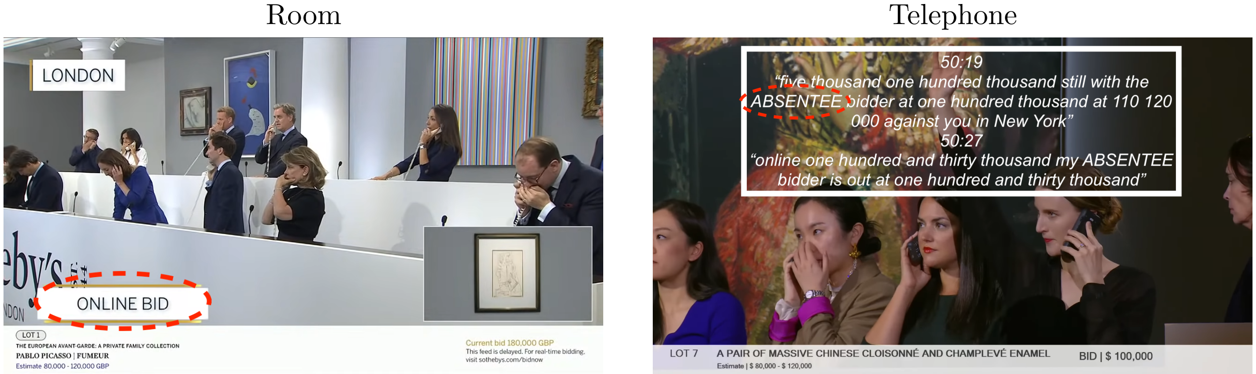


Lot 38, Christie's Hong Kong, Nov 2022 Lot 117, Sotheby's New York, Nov 2022

Figure 3: Example screenshots taken from Christie's and Sotheby's live auctions posted on YouTube. Red boxes highlight the relevant portions of the screen.



Room Telephone



Online Absentee

Figure 4: The four types of bidders in Sotheby's or Christie's live auctions.

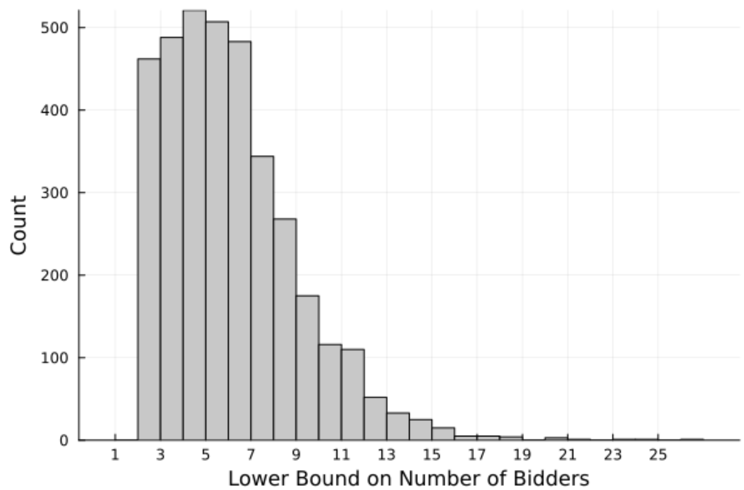


Figure 5: Distribution of lower bound on number of bidders across all auctions.

the bids accordingly.

6.2 Modern Art Sold in New York City

Our empirical analysis focuses on modern art sold in New York City. This is the highest subsample in our data both in terms of the number of auctions and generated revenue. To avoid outliers, we focus on the lots with transaction prices between \$100K and \$10.0M, which leaves us with 949 observations. We further condition on the price range, $[\$100\text{K}, \$1.0\text{M}]$ or $[\$1.0\text{M}, \$10.0\text{M}]$, roughly splitting the sample into one-third and two-thirds, and the number of bidders, $[2, 10]$ or $[6, 40]$ (i.e., $\underline{N} \leqslant 5$ vs $\underline{N} \geqslant 6$), splitting each of the subsamples approximately in half. Table 3 displays summary statistics for each of the selected price ranges. The median and average selling price are noticeably larger than the high estimates of the lot's value provided by the auction houses, confirming that the estimates are imprecise.

Examining the bidding data, we find that (i) there is no evidence of jump bids towards the end of the auctions; (ii) the difference between the two highest bids is small, relative to the size of the bids; and (iii) the auctioneers do not enforce the minimal bid increment guideline towards the end of the auction. Thus, the assumption that the transaction price reveals second-highest valuation appears plausible in our dataset. Each bidder's valuation for a given lot consists of a common component, representing the resale value, and a private component, representing consumption value. We assume that even after controlling for the art category, location, price range, and the number of bidders, the common component in the valuations is prevalent. Thus, we consider the bounding parameters $\underline{\rho}$ and $\overline{\rho}$ to values in the range $[0.75, 0.85]$. Normalizing the transaction price by the lot's high estimate, we consider two possible choices for the value of unsold good to the seller: $v_0 = 2/3$, corresponding to

Transaction price ∈ [$100K, $1.0M]; Number of lots: 287				
Variable	**Median**	**Mean**	**Std**	**[Min, Max]**
Transaction price/High estimate	1.39	1.83	1.29	[0.25, 6.35]
Second-highest bid	1.31	1.72	1.23	[0.24, 6.14]
Number of bids	10.00	11.42	7.40	[2.00, 39.00]
Number of bidders	5.00	5.44	2.50	[2.00, 13.00]
Low estimate/High estimate	0.67	0.69	0.05	[0.60, 0.78]
Transaction price ∈ [$1.0M, $10.0M]; Number of lots: 662				
Variable	**Median**	**Mean**	**Std**	**[Min, Max]**
Transaction price/High Estimate	1.15	1.41	0.85	[0.37, 6.72]
Second-highest bid	1.11	1.34	0.82	[0.35, 6.30]
Number of bids	10.00	11.75	8.04	[2.00, 76.00]
Number of bidders	5.00	5.98	2.91	[2.00, 20.00]
Low estimate/High estimate	0.67	0.68	0.05	[0.48, 0.84]

Table 3: Summary statistics for modern art auction lots in New York City.

the average low estimate, and $v_0 = 5/6$, corresponding the the midpoint between the low and high estimates.

Table 4 presents the results. With $v_0 = 5/6$ and restricted $\underline{\rho}, \overline{\rho}$, the estimated lower bounds on expected profit under Convex MMR reserves are consistently higher than the average realized profit, with the projected profit gains being larger for higher-priced lots with fewer bidders. In particular, for the lots within [$1.0M, $10.0M] price range with $N \in [2, 10]$ bidders, the lower bound on the profit gain under the Convex MMR reserve with $\underline{\rho} = \overline{\rho} = 0.8$ is 15.35%, which amounts to $572K per lot, or $17.1M per auction.[15] Pooling together all lots in that price range yields a more conservative lower bound on the profit gain of 3.85%, which amounts to $127K per lot or $3.8M per auction.[16] With $\underline{\rho} = 0$ and $\overline{\rho} = 1$, the lower bound on the profit under Convex MMR reserve may be substantially lower than the realized profit. With $v_0 = 2/3$ and restricted $\underline{\rho}, \overline{\rho}$, we find that the projected profit gains to be more modest with lower bounds occasionally being slightly below the realized profit. However, even in this arguably conservative scenario, the profit gains are generally expected to be positive.

[15]Recall that each auction consists of 30 lots, on average.

[16]Table 5 in the Appendix reports the results with lower values of $\underline{\rho}, \overline{\rho}$, which result in larger expected profit gains across the board.

Price Range	[$100K, $1.0M]			[$1.0M, $10.0M]		
Number of Bidders	$N \in$[2, 10]	$N \in$[6, 40]	$N \in$[2, 40]	$N \in$[2, 10]	$N \in$[6, 40]	$N \in$[2, 40]
$v_0 = 5/6$						
Avg High Estimate	$541,688	$316,470	$434,965	$3,811,117	$2,741,925	$3,291,057
Avg Realized Profit	0.545	1.502	0.998	0.228	0.937	0.573
Convex MMR reserve						
with $\underline{\rho} = 0.8, \overline{\rho} = 0.8$	1.20	1.36	1.25	1.12	1.05	1.20
– Bounds on profit	[0.587, 0.664]	[1.508, 1.530]	[1.034, 1.094]	[0.263, 0.337]	[0.951, 0.956]	[0.595, 0.666]
– 95% CI	[0.458, 0.789]	[1.302, 1.721]	[0.911, 1.220]	[0.216, 0.394]	[0.865, 1.041]	[0.539, 0.726]
Convex MMR reserve						
with $\underline{\rho} = 0.75, \overline{\rho} = 0.85$	1.14	1.32	1.14	1.31	1.02	1.12
– Bounds on profit	[0.592, 0.661]	[1.509, 1.534]	[1.034, 1.079]	[0.224, 0.363]	[0.949, 0.955]	[0.598, 0.656]
– 95% CI	[0.456, 0.782]	[1.310, 1.727]	[0.907, 1.196]	[0.136, 0.454]	[0.865, 1.038]	[0.544, 0.706]
Convex MMR reserve						
with $\underline{\rho} = 0.0, \overline{\rho} = 1$	1.73	0.84	2.02	0.84	0.85	0.84
– Bounds on profit	[0.414, 0.859]	[1.513, 1.514]	[0.759, 1.352]	[0.273, 0.277]	[0.944, 0.945]	[0.600, 0.602]
$v_0 = 2/3$						
Avg High Estimate	$541,688	$316,470	$434,965	$3,811,117	$2,741,925	$3,291,057
Avg Realized Profit	0.711	1.669	1.165	0.394	1.104	0.739
Convex MMR reserve						
with $\underline{\rho} = 0.8, \overline{\rho} = 0.8$	1.11	1.32	1.11	1.13	0.85	1.09
– Bounds on profit	[0.709, 0.786]	[1.661, 1.683]	[1.174, 1.221]	[0.328, 0.445]	[1.107, 1.108]	[0.722, 0.787]
– 95% CI	[0.580, 0.922]	[1.453, 1.872]	[1.057, 1.334]	[0.247, 0.511]	[1.018, 1.191]	[0.666, 0.854]
Convex MMR reserve						
with $\underline{\rho} = 0.75, \overline{\rho} = 0.85$	1.05	1.26	1.00	1.09	0.80	1.06
– Bounds on profit	[0.702, 0.777]	[1.668, 1.691]	[1.169, 1.206]	[0.342, 0.456]	[1.107, 1.107]	[0.720, 0.788]
– 95% CI	[0.567, 0.902]	[1.470, 1.891]	[1.038, 1.317]	[0.273, 0.535]	[1.017, 1.192]	[0.661, 0.856]
Convex MMR reserve						
with $\underline{\rho} = 0.0, \overline{\rho} = 1$	1.71	0.73	1.91	0.77	0.69	0.68
– Bounds on profit	[0.447, 0.961]	[1.674, 1.676]	[0.829, 1.444]	[0.393, 0.417]	[1.105, 1.106]	[0.745, 0.745]
Observations	151	136	287	340	322	662

Table 4: Bounds on expected profit under convex MMR reserve for Modern Art sold in New York City. Figures are scaled by the high estimate, and v_0 is set at 5/6 and 2/3.

7 Conclusion and Further Research

This paper proposed a novel approach to partial identification in open ascending auctions. We derived new bounds on the distribution of the highest valuation and seller's profit, bounding the degree of dependence between the valuations, and proposed a way to select a reserve price under ambiguity using the Min-Max-Regret criterion. The proposed approach is computationally simple and readily extends to settings with unknown number of bidders, as long as bounds on the number of bidders are available. We demonstrated that the resulting bounds are substantially tighter than those previously available in the literature, and the proposed reserve price performs well in practice. We applied the proposed methodology to a large new dataset of art auctions held by Christie's and Sotheby's and argued that higher reserve prices would lead to a sizable increase in profit.

Our analysis can be advanced in several directions. First, one may be interested in welfare aspects of the chosen auction format. Under the stated assumptions, bidders surplus is also identified from the marginal distributions of the two highest valuations and can be studied similarly to the expected profit. Second, the assumption that the transaction price reveals the second-highest valuation in the observed data can be relaxed to weaker bidding assumptions in the spirit of Haile and Tamer (2003). In such setting, bounds on the distribution of the highest valuation follow directly from the arguments above, but the analysis of optimal reserve prices becomes more nuanced: a coherent formulation of expected profit requires additional assumptions on bidding behavior in counterfactual auctions, and the Min-Max-Regret problems for selecting the reserve price become more complex. Finally, an interesting direction for further research is allowing for asymmetric bidders. While our bounding approach critically relies on the valuations being conditionally i.i.d., it appears possible to accommodate several "types" of symmetric bidders using similar ideas.

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A Proofs

A.1 Auxiliary Lemmas

Lemma A.1 (Properties of $\phi_N(\cdot)^N$)**.** *Consider a function $h_N : [0,1] \to [0,1]$ given by:*

$$h_N(t) = \phi_N(t)^N,$$

where $\phi_N : [0,1] \to [0,1]$ is defined implicitly via $t = N\phi_N(t)^{N-1} - (N-1)\phi_N(t)^N$. Then:

1. *$h'_N(t) = \frac{1}{N-1} \cdot \frac{\phi_N(t)}{1-\phi_N(t)}$, so h is strictly increasing.*
2. *$h''_N(t) = \frac{2}{N-1} \cdot \frac{1}{\binom{N}{2}\phi_N(t)^{N-2}(1-\phi_N(t))^2}$, so h is strictly convex, $h''(0) = h''(1) = \infty$.*
3. *$h'''_N(t) = \frac{N\phi_N(t)-(N-2)}{N(N-1)\phi_N(t)^{N-1}(1-\phi_N(t))} \cdot h''_N(t)$, so $h'_N(\cdot)$ changes curvature exactly once.*
4. *$f(t) = h_N(t)/t$ satisfies $\lim_{t\to 0} f(t) = 0$ and is strictly increasing on $(0,1)$.*
5. *$h_N(t)$ is increasing in N for all $t \in (0,1)$ and $N \geq 2$.*

Proof. Parts 1–3 follow immediately from taking derivatives. In Part 4, the first claim follows from the L'Hôpital's rule. For the second claim, notice that by definition of $\phi_N(t)$,

$$\frac{h(t)}{t} = \frac{\phi_N(t)^N}{N\phi_N(t) - (N-1)\phi_N(t)^N} = \frac{1}{\frac{N}{\phi_N(t)} - (N-1)},$$

which is a strictly increasing function of t.

For Part 5, note that ϕ_N is well-defined for any real $N \geqslant 2$, so we can rely on differentiation. Recall the identity:

$$\left(\alpha(N)^{\beta(N)}\right)'_N = \alpha(N)^{\beta(N)}\beta'(N)\log\alpha(N) + \alpha(N)^{\beta(N)-1}\beta(N)\alpha'(N).$$

Fix any $t \in (0,1)$ and suppress it from the notation for simplicity. Using the above identity:

$$(h_N)' = (\phi_N)^N \log\phi_N + N(\phi_N)^{N-1}(\phi_N)'. \tag{A.1}$$

To compute the derivative $(\phi_N)'$, recall the definition of ϕ_N:

$$t = N(\phi_N)^{N-1} - (N-1)(\phi_N)^N. \tag{A.2}$$

Differentiation both sides using the above identity and rearranging yields:

$$(\phi_N)' = \frac{(\phi_N)^N - (\phi_N)^{N-1} - t\log\phi_N}{N(N-1)(\phi_N)^{N-2}(1-\phi_N)}.$$

Plugging this in (A.1) and using (A.2), with some algebra:

$$\left(\phi^N\right)'_N = \frac{\phi_N}{(N-1)(1-\phi_N)}\left((\phi_N)^N - (\phi_N)^{N-1} - (\phi_N)^{N-1}\log\phi_N\right).$$

The right-hand side is non-negative since $\log(x) \leqslant x - 1$. ■

Lemma A.2 (Bounding Functions C_ρ Satisfying Assumption 2.3)**.** *The functions:*

1. $C_\rho(u) = \rho u + (1-\rho)u^2$
2. $C_\rho(u) = u^{2-\rho}$
3. $C_\rho(u) = P(\Phi(Z_1) \leqslant u, \Phi(Z_2) \leqslant u)$ *where* (Z_1, Z_2) *are jointly normal with mean zero, variances one, and covariance* ρ*;*

satisfy all requirements of Assumption 2.3.

Proof. Notice that $(C_\rho(u))' \geqslant \frac{C(u)}{u}$ is equivalent to $\frac{C_\rho(u)}{u}$ being increasing in u.

1. $C_0(u) = u^2$ and $C_1(u) = u$ is immediate. Further, $\frac{C_\rho(u)}{u} = \rho + (1-\rho)u$, which is increasing in u. Finally,

$$2\frac{u - C_\rho(u)}{1-u} = 2u(1-\rho) < \rho + 2u(1-\rho) = (C_\rho(u))',$$

2. $C_0(u) = u^2$ and $C_1(u) = u$ is immediate. Further, $\frac{C_\rho(u)}{u} = u^{1-\rho}$, which is increasing in u. Finally, consider:

$$\begin{aligned} g(u) &= (C_\rho(u))' - 2\frac{u - C_\rho(u)}{1-u} \\ &= (2-\rho)u^{1-\rho} - 2\frac{u - u^{2-\rho}}{1-u} \\ &= \frac{u^{1-\rho}}{1-u}\left\{2 - \rho + \rho u - 2u^\rho\right\}. \end{aligned}$$

It is straightforward to verify that the function in the curly brackets is non-negative.

3. As is well-known:

$$C_\rho(u) = 2\int_0^u \Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}\Phi^{-1}(t)\right)dt.$$

Hence, $C_0(u) = u^2$ and $C_1(u) = u$ is immediate. Further, since the integrand in the above display is non-negative and increasing in t,

$$C_\rho(u) \leqslant 2\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}\Phi^{-1}(u)\right)u$$

By the Leibniz rule,

$$(C_\rho(u))' = 2\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}\Phi^{-1}(u)\right),$$

so the inequality in the preceding display states $(C_\rho(u))' \geqslant \frac{C_\rho(u)}{u}$. It remains to show $(C_\rho(u))' \geqslant 2\frac{u-C_\rho(u)}{1-u}$. Consider two cases.

(i) Suppose $u \leqslant \frac{1}{2}$ so that $\Phi^{-1}(u) \leqslant 0$. Then,

$$(C_\rho(u))' \geqslant 2u = 2\frac{u-u^2}{1-u} \geqslant 2\frac{u-C_\rho(u)}{1-u},$$

where the first inequality follows from $\sqrt{\frac{1-\rho}{1+\rho}}\Phi^{-1}(u) \geqslant \Phi^{-1}(u)$ and the second from $C_\rho(u) \geqslant u^2$.

(ii) Suppose $u \geqslant \frac{1}{2}$ so that $\Phi^{-1}(u) \geqslant 0$. Then, $(C_\rho(u))' \geqslant 2\Phi(0) = 1$, and

$$\begin{aligned} C_\rho(u) &= \int_0^{1/2} (C_\rho(t))'dt + \int_{1/2}^u (C_\rho(t))'dt \\ &\geqslant \int_0^{1/2} 2tdt + \int_{1/2}^u 1dt \\ &= u - \frac{1}{4}. \end{aligned}$$

Therefore:

$$2\frac{u-C_\rho(u)}{1-u} \leqslant \frac{1}{2}\cdot\frac{1}{1-u} \leqslant 1 = (C_\rho(u))'$$

where the last inequality holds since $u \geqslant \frac{1}{2}$.

■

A.2 Proposition 1

Proof of Proposition 1. The upper bound on $F_{N:N}(v)$ is trivial and binds in the case of pure common values. For the lower bound, Lemma A.1 establishes that $u \mapsto \phi_N(u)^N$ is strictly convex. Thus:

$$\begin{aligned} P(V_{N:N} \leqslant v) &= \mathbb{E}[P(V_{N:N} \leqslant v \,|\, U)] \\ &= \mathbb{E}[P(V_i \leqslant v \,|\, U)^N] \\ &\overset{(a)}{=} \mathbb{E}[\phi_N(P(V_{N-1:N} \leqslant v \,|\, U))^N] \\ &\overset{(b)}{\geqslant} \phi_N(F_{N-1:N}(v))^N, \end{aligned}$$

where (a) follows from the definition of $\phi_N(\cdot)$, and (b) from convexity of $t \mapsto \phi_N(t)^N$, Jensen's inequality, and the law of iterated expectations. ■

A.3 Theorem 1

Proof of Theorem 1. In the main part of the proof, we show that the stated optimization problems lead to pointwise sharp bounds on $F_{N:N}(v)$. Lemmas A.3 and A.4 show that the bounds are monotone in $F_{N-1:N}(v)$ and thus are plausibly sharp in the functional sense.

For the lower bound, consider the generalized moment problem:

$$\inf_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[\phi_N(T)^N] \;\middle|\; \mathbb{E}_P[T] = \mu, c_1 \leqslant \mathbb{V}ar_P(T) \leqslant c_2 \right\} \tag{A.3}$$

where $\mathcal{M}[0,1]$ denotes the set of all probability measures supported on $[0,1]$, $\mu \in [0,1]$ and $0 \leqslant c_1 \leqslant c_2 \leqslant \mu(1-\mu)$. Denote $h(t) = \phi_N(t)^N$. Lemma A.1 establishes that $h(t)$ is strictly increasing, strictly convex, and that $h'(t)$ changes curvature exactly once and satisfies $h''(0) = h''(1) = \infty$. Since $h(t)$ is convex, the variance constraint $\mathbb{V}ar_P(T) = c_1$ must be binding, so the problem is:

$$\inf_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[h(T)] \,|\, \mathbb{E}_P[T] = \mu, \mathbb{E}_P[T^2] = \mu^2 + c_1 \right\}. \tag{A.4}$$

When $c_1 = 0$, by Jensen's inequality, the infimum is attained by the distribution $P(T = \mu) = 1$ and equals $h(\mu)$. When $c_1 = \mu(1-\mu)$, the only feasible distribution on $[0,1]$ is the Bernoulli distribution with $P(T = 1) = \mu$. In this case, the infimum is equal to μ. It remains to consider $c_1 \in (0, \mu(1-\mu))$.

It is well known that it suffices to consider distributions P with at most three support points (e.g., Theorem 1 in Kemperman, 1968). We first show that, due to the specific shape of the function $h(\cdot)$ and its derivative, it suffices to consider distributions with two support points. Denote $g_1(t) = t$, $g_2(t) = t^2$, and $g(t) = (g_1(t), g_2(t))$ all defined on $\mathcal{T} = [0,1]$. Note that $V = \text{Conv}(g(\mathcal{T})) = \{(z_1, z_2) \in [0,1]^2 : z_1^2 \leqslant z_2 \leqslant z_1\}$, and the point $y = (\mu, \mu^2 + c_1) \in \text{Int(V)}$. Denote $D^* = \{d^* = (d_0, d_1, d_2) \in \mathbb{R}^3 : d_0 + d_1 g_1(t) + d_2 g_2(t) \leqslant h(t) \text{ for all } t \in [0,1]\}$, and, for a given $d^* \in D^*$, let $B(d^*) = \{z = g(t) : d_0 + d_1 g_1(t) + d_2 g_2(t) = h(t) \text{ for some } t \in T\}$. By Theorem 5 and the following remark in Kemperman (1968), for every $y \in \text{Int}(V)$, there exists a d^* for which $y \in \text{Conv}(B(d^*))$. By Theorem 4 of the same paper, such y can be expressed as $y = \sum_{j=1}^m p_j g(t_j)$ for some $g(t_j) \in B(d^*)$, and the minimal value of the moment problem is given by $\sum_{j=1}^m p_j h(t_j)$. In the problem under consideration, $B(d^*)$ can contain at most two points. Suppose $k(t) = d_0 + d_1 t + d_2 t^2 \leqslant h(t)$, for all $t \in \mathcal{T}$, with equality for some $t_1 < \ldots < t_m$. Then: (i) if $t_1 = 0$, there can be at most one other $t_2 \in (0,1)$; (ii) if all $t_j \in (0,1)$, it must be the case that the line $k'(t) = d_1 + 2d_2 t$ intersects the curve $h'(t)$ from above at each t_j.[17] Indeed, case (i) follows from direct computation, and case (ii) from a simple geometric fact that since $h'(t)$ changes curvature only once (concave then convex) and satisfies $h''(0) = h''(1) = \infty$, there are at most two interior intersections of $d_1 + 2d_2 t$ and $h'(t)$. Thus, the set $B(d^*)$ contains at most two poins, so to solve (A.4), it suffices to consider distributions P with two support points.

For some $0 \leqslant a < b \leqslant 1$ and $p \in [0,1]$, consider the distribution P with $P(X = a) = p$ and $P(X = b) = 1 - p$. The constraints are:

$$\begin{cases} ap + b(1-p) = \mu \\ a^2 p + b^2(1-p) = \mu^2 + c_1 \end{cases} \implies \begin{cases} a(p) = \mu - \sqrt{\frac{1-p}{p} c_1} \\ b(p) = \mu + \sqrt{\frac{p}{1-p} c_1}, \end{cases}$$

and p must be subject to $a, b \in [0,1]$. When $p = 0$ or $p = 1$, $a = b = \mu$ which violates the variance constraint for $c_1 > 0$. Thus, the problem is:

$$\begin{aligned} &\min_{p \in (0,1)} \left\{ h\left(\mu - \sqrt{\tfrac{1-p}{p} c_1}\right) p + h\left(\mu + \sqrt{\tfrac{p}{1-p} c_1}\right)(1-p) \right\} \\ &\text{s.t. } a(p), b(p) \in [0,1] \end{aligned}$$

Leting $s = \sqrt{\frac{1-p}{p} c_1}$ so that $p = \frac{c_1}{c_1 + s^2}$ yields an equivalent formulation:

$$\min_{s \in [\frac{c_1}{1-\mu}, \mu]} \left\{ h(\mu - s) \frac{c_1}{c_1 + s^2} + h\left(\mu + \frac{c_1}{s}\right) \frac{s^2}{c_1 + s^2} \right\}.$$

By direct computation, the objective function is seen to be strictly convex on the feasible

[17] For $t_j \in (0,1)$ it means that $k'(t) > h'(t)$ immediately before t_j and $k'(t) < h'(t)$ immediately after. For $t_j \in \{0,1\}$ only one of the two preceding inequalities is required to hold.

set.

Next, consider the generalized moment problem:

$$\sup_{P\in\mathcal{M}[0,1]} \left\{ \mathbb{E}_P[h(T)] \,\middle|\, \mathbb{E}_P[T] = \mu, c_1 \leqslant \mathbb{V}ar_P(T) \leqslant c_2 \right\},$$

or, equivalently,

$$-\inf_{P\in\mathcal{M}[0,1]} \left\{ \mathbb{E}_P[-h(T)] \,\middle|\, \mathbb{E}_P[T] = \mu, c_1 \leqslant \mathbb{V}ar_P(T) \leqslant c_2 \right\}.$$

Since $f(t) = -h(t)$ is concave, the variance constraint $\mathbb{V}ar_P(T) = c_2$ must be binding, so the problem is:

$$-\inf_{P\in\mathcal{M}[0,1]} \left\{\mathbb{E}_P[f(T)] \,|\, \mathbb{E}_P[T] = \mu, \mathbb{E}_P[T^2] = \mu^2 + c_2\right\}, \tag{A.5}$$

with $\mu \in [0, 1]$ and $c_2 \in [0, \mu(1-\mu)]$. When $c_2 = 0$, by the Edmundson-Mandansky inequality, the infimum is attained by the distribution $P(T = \mu) = 1$ and equals $f(\mu)$. When $c_2 = \mu(1-\mu)$, the only feasible distribution on $[0, 1]$ is the Bernoulli distribution with $P(T = 1) = \mu$. In this case, the infimum is equal to μ. It remains to consider $c_2 \in (0, \mu(1-\mu))$.

Using the idea and notation from the first part of the proof, suppose $k(t) = d_0 + d_1 t + d_2 t^2 \leqslant f(t)$ for all $t \in \mathcal{T}$ with equality for some $t_1 < \ldots < t_m$. Then: at each $t_j \in (0, 1)$, the line $k'(t)$ must intersect $f'(t)$ from above; if $t_1 = 0$, it must be $k'(0) \leqslant f'(0)$; and if $t_m = 1$, it must be that $k'(1) \geqslant f'(1)$. The function $f'(t)$ changes curvature exactly once (convex then concave) and satisfies $f'(0) = 0$ and $f'(1) = -\infty$. Thus, there can be at most one interior t_j satisfying the requirement above. By Theorem 4 of Kemperman (1968), it suffices to consider distributions with tree support points: 0, 1, and $t \in (0, 1)$. Letting $P(T = t) = q$, $P(T = 1) = p$, and $P(T = 0) = 1 - p - q$, the constraints in (A.5) become:

$$\begin{cases} p + qt = \mu \\ p + qt^2 = \mu^2 + c_2 \end{cases} \implies \begin{cases} q(t) = \frac{\mu(1-\mu)-c_2}{t(1-t)} \\ p(t) = \mu - q(t)\cdot t \end{cases},$$

subject to $q(t), p(t) \geqslant 0$ and $q(t) + p(t) \leqslant 1$. Plugging this into (A.5) and rearranging yields:

$$\max_{t\in\left[\mu-\frac{c_2}{1-\mu}, \mu+\frac{c_2}{\mu}\right]} \left\{\mu - \frac{\mu(1-\mu) - c_2}{t(1-t)}(t - h(t))\right\}.$$

■

A.3.1 Properties of Bounds in Theorem 1

Lemma A.3 (Properties of the Lower Bound in Theorem 1)**.** *Consider the function:*

$$f(\mu) = \min_{s\in[\frac{c_1}{1-\mu},\mu]} \left\{ \phi_N(\mu-s)^N \frac{c_1}{c_1+s^2} + \phi_N\left(\mu+\frac{c_1}{s}\right)^N \frac{s^2}{c_1+s^2} \right\}$$

with $c_1 = C_\rho(\mu,\mu) - \mu^2$ *with* C_ρ *satisfying the requirements in Assumption 2.3. Then,* $\mu \mapsto f(\mu)$ *is non-decreasing.*

Proof. We suppress the dependence of c_1 on μ to simplify the notation. Denoting the objective function in the optimization problem by $r(s;\mu,c_1)$,

$$\frac{\partial r(s;\mu,c_1)}{\partial s} = \frac{2c_1 s}{(c_1+s^2)} \left\{ h\left(\mu+\frac{c_1}{s}\right) - h(\mu-s) \right\} - \frac{c_1}{c_1+s^2}\left\{ h'(\mu-s) + h'\left(\mu+\frac{c_1}{s}\right)\right\}.$$

First, note that as $s \to \frac{c_1}{1-\mu}$, we have $\mu - c_1/s \to 1$ and thus $h'(\mu - c_1/s) \to +\infty$, by Lemma A.1. Since all other terms in the above expression stay finite, $r'(s;\mu,c_1) \to -\infty$. Thus, the constraint $s \geqslant c_1/(1-\mu)$ never binds, and it suffices to consider two cases: (i) $s = \mu$ and (ii) $s \in (c_1/(1-\mu), \mu)$. In case (i), the lower bound is given by:

$$f(\mu) = \frac{\mu^2}{c_1+\mu^2} h\left(\mu + \frac{c_1}{\mu}\right) = \mu \frac{h(\frac{C_\rho(\mu,\mu)}{\mu})}{\frac{C_\rho(\mu,\mu)}{\mu}}.$$

The function $u \mapsto h(u)/u$ is increasing, by Lemma A.1, and $C_\rho(\mu,\mu)/\mu$ is increasing by Assumption 2.3, so $f(\mu)$ is increasing. For case (ii), by the Envelope Theorem:

$$f'(\mu) = \left.\frac{\partial r(s,\mu,c_1)}{\partial \mu}\right|_{s=s^*(\mu)} + \left.\frac{\partial r(s;\mu,c_1)}{\partial c_1}\right|_{s=s^*(\mu)} \cdot \frac{\partial c_1}{\partial \mu},$$

where $s^*(\mu)$ solves the first order condition $\frac{\partial r(s;\mu,c_1)}{\partial s} = 0$,

$$\frac{\partial r(s,\mu,c_1)}{\partial \mu} = \frac{c_1}{c_1+s^2} h'(\mu-s) + \frac{s^2}{c_1+s^2} h'\left(\mu+\frac{c_1}{s}\right),$$

and

$$\frac{\partial r(s;\mu,c_1)}{\partial c_1} = \frac{s^2}{(c_1+s^2)^2}\left\{ h(\mu-s) - h\left(\mu+\frac{c_1}{s}\right)\right\} + \frac{s}{c_1+s^2} h'\left(\mu+\frac{c_1}{s}\right). \tag{A.6}$$

Thus, the condition $f'(\mu) \geqslant 0$ is equivalent to:

$$\frac{\partial c_1(\mu)}{\partial \mu} \geqslant - \left. \frac{\frac{\partial r(s,\mu,c_1)}{\partial \mu}}{\frac{\partial r(s;\mu,c_1)}{\partial c_1}} \right|_{s=s^*(\mu)} \tag{A.7}$$

From the first order condition, at $s = s^*(\mu)$,

$$h\left(\mu + \frac{c_1}{s}\right) - h(\mu - s) = \frac{c_1 + s^2}{2} \left\{ h'(\mu - s) + h'(\mu + \frac{c_1}{s}) \right\}.$$

Plugging the above in (A.6) and the result in (A.7), and applying the Mean Value Theorem yields:

$$\begin{aligned} - \left. \frac{\frac{\partial r(s,\mu,c_1)}{\partial \mu}}{\frac{\partial r(s;\mu,c_1)}{\partial c_1}} \right|_{s=s^*(\mu)} &= -2 \left. \frac{h'(\mu - s) + \frac{s^2}{c_1+s^2}\{h'(\mu + \frac{c_1}{s}) - h'(\mu - s)\}}{\frac{s}{c_1+s^2}\{h'(\mu + \frac{c_1}{s}) - h'(\mu - s)\}} \right|_{s=s^*(\mu)} \\ &= -2 \left. \frac{h'(\mu - s) + sh''(\tilde{\mu})}{h''(\tilde{\mu})} \right|_{s=s^*(\mu)} \\ &= -2(s^*(\mu) + \tfrac{h'(\mu - s^*(\mu))}{h''(\tilde{\mu})}), \end{aligned}$$

for some $\tilde{\mu} \in [\mu - s^*(\mu), \mu + \frac{c_1}{s^*(\mu)}]$. Since both $h'(\cdot)$ and $h''(\cdot)$ are positive, and $s^*(\mu) \geqslant c_1/(1-\mu)$ a sufficient condition for the lower bound to increase in μ is:

$$\frac{\partial c_1(\mu)}{\partial \mu} \geqslant -2\frac{c_1(\mu)}{1-\mu}.$$

Plugging in the expression for $c_1(\mu)$ and simplifying yields:

$$(C_\rho(\mu, \mu))' \geqslant 2 \cdot \frac{\mu - C_\rho(\mu, \mu)}{1 - \mu}.$$

■

Lemma A.4 (Properties of the Upper Bound in Theorem 1)**.** *Consider the function:*

$$f(\mu) = \max_{s \in \left[\mu - \frac{c_2}{1-\mu}, \mu + \frac{c_2}{\mu}\right]} \left\{ \mu - \frac{\mu(1-\mu) - c_2}{s(1-s)}(s - h(s)) \right\}.$$

where $h(s) = \phi_N(s)^N$ and $c_2 = C_\rho(\mu, \mu) - \mu^2$ with C_ρ satisfying the requirements in Assumption 2.3. Then, $\mu \mapsto f(\mu)$ is non-decreasing.

Proof. Let $s^*(\mu)$ denote the argmax. There are three possible cases.

(i) $s^*(\mu) = \mu - \frac{c_2}{1-\mu} = \frac{\mu - C_\rho(\mu,\mu)}{1-\mu}$. Plugging this in the objective function and simplifying yields:

$$f(\mu) = \underbrace{\frac{C_\rho(\mu,\mu) - \mu^2}{C_\rho(\mu,\mu) - \mu^2 + (1-\mu)^2}}_{\alpha(\mu)} \cdot 1 + \underbrace{\frac{(1-\mu)^2}{C_\rho(\mu,\mu) - \mu^2 + (1-\mu)^2}}_{1-\alpha(\mu)} h(s^*(\mu)).$$

Since $h(s^*(\mu)) \leqslant 1$, for all μ, it suffices to show that $\alpha(\mu)$ is increasing. In turn, this is equivalent to

$$\beta(\mu) = \frac{(1-\mu)^2}{C_\rho(\mu,\mu) - \mu^2}$$

being decreasing. By direct calculation, $\beta'(\mu) \leqslant 0$ is equivalent to:

$$(C_\rho(\mu,\mu))' \geqslant 2\frac{\mu - C_\rho(\mu,\mu)}{1-\mu},$$

which is one of the conditions in Assumption 2.3.

(ii) $s^*(\mu)$ is interior. Denoting the objective function in the optimization problem by $r(s;\mu,c_2)$, by the Envelope Theorem:

$$\begin{aligned} f'(\mu) &= \left.\frac{\partial r(s,\mu,c_2)}{\partial \mu}\right|_{s=s^*(\mu)} + \left.\frac{\partial r(s;\mu,c_2)}{\partial c_2}\right|_{s=s^*(\mu)} \cdot \frac{\partial (C_\rho(\mu,\mu) - \mu^2)}{\partial \mu} \\ &= 1 - (1-2\mu)\frac{s^* - h(s^*)}{s^* - s^{*2}} + \frac{s^* - h(s^*)}{s^* - s^{*2}} \cdot \left(\frac{\partial C_\rho(\mu,\mu)}{\partial \mu} - 2\mu\right) \\ &= 1 + t(s^*)\left(\frac{\partial C_\rho(\mu,\mu)}{\partial \mu} - 1\right), \end{aligned}$$

where $t(s^*) \equiv \frac{s^* - h(s^*)}{s^* - s^{*2}}$. Notice the solution s^* does not depend on μ and corresponds to the minimum of the function $t(s)$. Since $\frac{\partial C_\rho(\mu,\mu)}{\partial \mu} \geqslant 0$, to conclude that $f'(\mu) \geqslant 0$ it suffices to show that $t(s^*) \in [0,1]$. Since $h(s) \leqslant s$, we have that $t(s) \geqslant 0$ for all s. Writing $t(s) = 1 + \frac{s^2 - h(s)}{s - s^2} \leqslant 1$, it suffices to show $s^2 - h(s) \leqslant 0$ for some s. By the Taylor expansion, $h(s) = \frac{h''(\tilde{s})}{2}s^2$, for some $\tilde{s} \in (0,s)$, and, by Lemma A.1, $h''(\tilde{s}) \to +\infty$ as $\tilde{s} \to 0$. Thus, for s close to zero, $s^2 - h(s) < 0$, and the result follows.

(iii) $s^*(\mu) = \mu + \frac{c_2}{\mu} = \frac{C_\rho(\mu,\mu)}{\mu}$. Plugging this in the objective function and simplifying yields:

$$f(\mu) = \mu \frac{h\left(\frac{C(\mu,\mu)}{\mu}\right)}{\left(\frac{C(\mu,\mu)}{\mu}\right)}.$$

Since $\mu \mapsto \frac{C(\mu,\mu)}{\mu}$ is increasing, by Assumption 2.3, and $t \mapsto \frac{h(t)}{t}$ is increasing, by Lemma A.1, it follows that $f(\mu)$ is increasing.

■

A.4 Theorem 3

Proof of Theorem 3. Starting with Equation (6) and treating the lower and the upper bounds separately, the proof proceeds exactly like that of Theorem 1.

■

A.5 Min-Max-Regret Theorems 5 and 6

The following auxiliary lemma establishes that, given a vector of order statistics, it is without loss of generality to assume that the underlying distribution is exchangeable.

Lemma A.5 (Symmetrization)**.** *Let $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ be a random vector with an arbitrary joint distribution. Let $X^{1:n} = (X_{1:n}, \dots, X_{n:n})$ denote the vector of order statistics, where $X_{j:n}$ is the j-th smallest of $(X_1, \dots, X_n)$. There exists a random vector $Y = (Y_1, \dots, Y_n) \in \mathbb{R}^n$ such that: (1) Y is exchangeable; and (2) $Y^{1:n} = X^{1:n}$ almost surely.*

Proof. Let Π denote the set of all permutations $p : \{1, \dots, n\} \to \{1, \dots, n\}$, and π be drawn uniformly in Π, i.e., $P(\pi = p) = 1/n!$ for all $p \in \Pi$. Define $(Y_j)_{j=1}^n = (X_{\pi(j)})_{j=1}^n$. Then:

$$P(Y_1 \leqslant y_1, \dots, Y_n \leqslant y_n) = \frac{1}{n!} \sum_{p \in \Pi} P(X_{p(1)} \leqslant y_1, \dots, X_{p(n)} \leqslant y_n).$$

The summation in the right hand side includes all possible events of the form $\{X_{j_1} \leqslant y_1, \dots, X_{j_n} \leqslant y_n\}$, so an arbitrary permutation of $\{y_1, \dots, y_n\}$ changes the order of summands but not the value of the sum in the above display. Therefore, for any permutation p,

$$P(Y_1 \leqslant y_{p(1)}, \dots, Y_n \leqslant y_{p(n)}) = P(Y_1 \leqslant y_1, \dots, Y_n \leqslant y_n).$$

Since rearranging the elements of X does not affect the order statistics, we have $Y^{1:n} = X^{1:n}$ for all realizations of π, so that $P(Y^{1:n} = X^{1:n}) = 1$. ■

A.5.1 Theorem 5

Proof of Theorem 5. Let F denote $F_{N-1:N}$ and F^* denote $F_{N:N}$, for simplicity. Let $\pi(r;F^*)$ denote the expected profit function in (1), and $\mathcal{F}$ the set of all admissible F^*. Fix some reserve $r \geqslant v_0$ and consider an alternative $\tilde{r} \geqslant v_0$. Denote $D(\tilde{r},r;F^*) = \pi(\tilde{r};F^*) - \pi(r;F^*)$, which is

$$D(\tilde{r},r;F^*) = \int_{\tilde{r}}^{r} (1 - F(v))dv + \tilde{r} - r + \{F^*(r)(r - v_0) - F^*(\tilde{r})(\tilde{r} - v_0)\}.$$

The maximum regret is given by $R(r) = \max\{R_1(r), R_2(r)\}$, where

$$R_1(r) = \sup_{F_1} \sup_{\tilde{r}<r} D(\tilde{r},r;F^*).$$

$$R_2(r) = \sup_{F_1} \sup_{\tilde{r}>r} D(\tilde{r},r;F^*).$$

Considering $D(\tilde{r},r;F^*)$, the supremum $R_1(r)$ is attained by (i) maximizing $F^*(r)$; and (ii) minimizing $F^*(\tilde{r})$ for all $\tilde{r} < r$. Evidently, both conditions are satisfied by the CDF $F^*_{1,r}(\cdot)$ in the statement of the theorem. Similarly, the supremum $R_2(r)$ is attained by: (i) maximizing $F^*(r)$; and (ii) minimizing $F^*(\tilde{r})$ for all $\tilde{r} > r$. Condition (i) is achieved by setting $F^*(r) = \overline{F}(r)$. Since F^* must be non-decreasing, condition (ii) is achieved by setting $F^*(v) = \overline{F}(r)$ for all $v \in [r, \underline{F}^{-1}(\overline{F}(r))]$ and to $F^*(v) = \underline{F}(v)$ afterwards. Both of these conditions are satisfied by the CDF $F^*_{2,r}(\cdot)$ in the statement of the theorem. Hence, the result follows. ■

A.5.2 Theorem 6

Proof of Theorem 6. The argument is essentially the same as in Theorem 5. The supremum $R_1(r)$ is attained by (i) maximizing $F_1(r)$; and (ii) minimizing $F_1(\tilde{r})$ for all $\tilde{r} < r$. Since it must be that $F_1(v) = g_1(F_2(v))$ for a convex function g_1, conditions (i) and (ii) can only be attained by defining g_1 as in the statement of the Theorem. The middle part of g_1 is restricted to be linear precisely to preserve convexity while minimizing $F_1(\tilde{r})$ for all $r \leqslant \tilde{r}$. A similar argument applies for the supremum $R_2(r)$ and g_2. ■

Finally, we show the following auxiliary result. Its' proof additionally shows that $V_{N-1:N}$ in the statement of Assumption 3.2 can be replaced with $V_{N-1:N-1}$.

Lemma A.6 (CDFs of Order Statistics under Monotone Likelihood Ratio)**.** *Let Assumptions 3.2 (i) and (iv) hold. Then,*

$$F_{N:N} = g_N(F_{N-1:N})$$

for some increasing convex function $g_N : [0,1] \to [0,1]$.

Proof. By exchangeability,

$$\begin{aligned} F_{N-1:N}(v) &= N \cdot F_{N-1:N-1}(v) - (N-1) \cdot F_{N:N}(v) \\ &= N \cdot C_{N-1}(F(v)) - (N-1) \cdot C_N(F(v)), \end{aligned}$$

and

$$F_{N:N}(v) = C_N(F(v)),$$

where $C_J : [0,1] \to [0,1]$ is defined as $C_J(u) = \tilde{C}_J(u, \ldots, u)$ where $\tilde{C}_J$ is the copula function of $(V_1, \ldots, V_J)$, for $J \in \{N-1, N\}$. Denote $f(u) = NC_{N-1}(u) - (N-1)C_N(u)$ and notice this function is strictly increasing. It follows that:

$$F_{N:N}(v) = C_N(f^{-1}(F_{N-1:N}(v))) \equiv g(F_{N-1:N}(v)).$$

By direct calculation:

$$g''(u) = \frac{C_N''(s_u)f'(s_u) - C_N'(s_u)f''(s_u)}{f'(s_u)^2},$$

where $s_u = f^{-1}(u)$. Plugging in the expressions for $f'(s_u)$ and $f''(s_u)$ and simplifying yields that $g''(u) \geqslant 0$ for all $u \in [0,1]$ holds if and only if:

$$\frac{C_N''(s)}{C_N'(s)} \geqslant \frac{C_{N-1}''(s)}{C_{N-1}'(s)},$$

for all $s \in [0,1]$. Denoting $c_J(s) = C_J'(s)$ for $J \in \{N-1, N\}$, the above is equivalent to:

$$\frac{\partial}{\partial s}\left(\frac{c_N(s)}{c_{N-1}(s)}\right) \geqslant 0.$$

Since $c_N(v) = f_{V_{N:N}}(v)$ and $f_{V_{N-1:N}}(v) = Nc_{N-1}(v) - (N-1)f_{V_{N:N}}(v)$, it follows that:

$$\frac{f_{V_{N-1:N}}(v)}{f_{V_{N:N}}(v)} = N\frac{c_{N-1}(v)}{c_N(v)} - (N-1),$$

so the ratio $\frac{f_{V_{N:N}}(v)}{f_{V_{N-1:N}}(v)}$ is increasing if and only if $\frac{c_N(s)}{c_{N-1}(s)}$ is, which concludes the proof. ■

B Estimation

To simplify notation, we denote $F(v) = F_{N-1:N}(v)$ throughout. The exposition below assumes that N is known and fixed and $X = \varnothing$. To account for varying N and discrete X, it suffices to modify the estimator $\hat{F}$ and its asymptotic distribution.

B.1 Profit Bounds

Let $\hat{F}(v) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(P_i \leqslant v)$ denote the sample analog estimator for F. Its' pointwise asymptotic distribution is

$$\sqrt{n}(\hat{F}(v) - F(v)) \quad \to_d \quad N(0, V_F(v)),$$

where $V_F(v) = F(v)(1 - F(v)$. The profit bounds from Theorem 2 are estimated as

$$\begin{aligned} \hat{\underline{\pi}}(r) &= \frac{1}{n}\sum_{i=1}^{n}\max(r, P_i) - v_0 - (r - v_0)\overline{\psi}(\hat{F}(r)) \\ \hat{\overline{\pi}}(r) &= \frac{1}{n}\sum_{i=1}^{n}\max(r, P_i) - v_0 - (r - v_0)\underline{\psi}(\hat{F}(r)). \end{aligned} \tag{B.1}$$

Using delta-method, the asymptotic distributions are

$$\begin{aligned} \sqrt{n}(\hat{\underline{\pi}}(r) - \underline{\pi}(r)) &\quad \to_d \quad N(0, V_{\underline{\pi}}(r)); \\ \sqrt{n}(\hat{\overline{\pi}}(r) - \overline{\pi}(r)) &\quad \to_d \quad N(0, V_{\overline{\pi}}(r)), \end{aligned}$$

where

$$\begin{aligned} V_{\underline{\pi}}(r) &= \mathbb{V}ar(\max(r, P)) + (r - v_0)^2\overline{\psi}(F(r))^2 V_F(r) - 2(r - v_0)\overline{\psi}'(F(r))(r - \mathbb{E}[\max(r, P)])F(r); \\ V_{\overline{\pi}}(r) &= \mathbb{V}ar(\max(r, P)) + (r - v_0)^2\underline{\psi}(F(r))^2 V_F(r) - 2(r - v_0)\underline{\psi}'(F(r))(r - \mathbb{E}[\max(r, P)])F(r). \end{aligned}$$

Under standard regularity conditions, the plug-in variance estimators $\hat{V}_{\underline{\pi}}(r)$ and $\hat{V}_{\overline{\pi}}(r)$ are consistent.

B.2 MMR Reserve Price

Let $\hat{r}$ be defined in Equation (13). Suppose (i) The set of relevant reserve prices is $\mathcal{R} = [\underline{r}, \overline{r}]$; (ii) $\hat{F}_{N-1:N}$ is consistent for $F_{N-1:N}$ uniformly over $\mathcal{R}$; (ii) the MMR solution $r^* \in \mathcal{R}$ is unique. Then, by the standard arguments, $\hat{r}$ is consistent for r. To obtain the limiting distribution, suppose, for concreteness, that $\hat{F}_{N-1:N}$ is $\sqrt{n}$-consistent for $F_{N-1:N}$. By the mean-value theorem,

$$\hat{\Delta}(\hat{r}) - \hat{\Delta}(r^*) = \hat{\Delta}'(\tilde{r})(\hat{r} - r^*)$$

for some $\tilde{r}$ between $\hat{r}$ and r^*. Suppose that: (i) $\sqrt{n}\hat{\Delta}(\hat{r}) = o_P(1)$; (ii) $\sqrt{n}\hat{\Delta}(r^*) \to_d N(0, V_\Delta)$; (iii) $\sup_{r\in\mathcal{R}} |\hat{\Delta}'(r) - \Delta'(r)| = o_P(1)$; and (iv) $\Delta'(r^*) > 0$. Condition (i) allows for approximate solution $\hat{\Delta}(\hat{r}) \approx 0$; Condition (ii) can be established using functional delta-method, viewing Δ as a functional of F, although the expression for V_Δ is cumbersome; Condition (iii) follows from continuity of $F \mapsto \Delta'(r; F)$ with respect to the sup norms; and Condition (iv) ensures that the first-order mean-value expansion of $\hat{\Delta}(\cdot)$ is non-degenerate. By Slutsky's theorem,

$$\sqrt{n}(\hat{r} - r^*) \quad \to_d \quad N\left(0,\ \frac{V_\Delta}{\Delta'(r^*)^2}\right). \tag{B.2}$$

B.3 Expected Profit Under MMR Optimal Reserve Price

Consider the expansion from equation (16):

$$\begin{aligned}
\sqrt{n}(\hat{\underline{\pi}}^* - \underline{\pi}^*) \ &= \{\sqrt{n}(\hat{\underline{\pi}}(\hat{r}) - \underline{\pi}(\hat{r})) - \sqrt{n}(\hat{\underline{\pi}}(r^*) - \underline{\pi}(r^*))\} && (I) \\
&+ \sqrt{n}(\hat{\underline{\pi}}(r^*) - \underline{\pi}(r^*)) && (II) \\
&+ \sqrt{n}(\underline{\pi}(\hat{r}) - \underline{\pi}(r^*)). && (III)
\end{aligned} \tag{B.3}$$

Term (I) can be written as

$$\begin{aligned}
(I) \ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\max(\hat{r}, P_i) - \mathbb{E}_P[\max(\hat{r}, P)]\} - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\max(r^*, P_i) - \mathbb{E}_P[\max(r^*, P)]\} && (A) \\
&+ \sqrt{n}\left(\{\underline{\psi}(\hat{F}(\hat{r})) - \underline{\psi}(F(\hat{r}))\} - \{\underline{\psi}(\hat{F}(r^*)) - \underline{\psi}(F(r^*))\}\right), && (B)
\end{aligned}$$

where $\mathbb{E}_P[\,\cdot\,]$ denotes the expectation with respect to P only. Term (A) takes the form

$$(A) = \mathbb{G}_n(\hat{f} - f)$$

where $\mathbb{G}_n(h) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} h(P_i) - \mathbb{E}[h(P)]$ denotes the empirical process, $f(P) = \max(r, P)$, and $\hat{f}(P) = \max(\hat{r}, P)$. Notice $\mathcal{F} = \{f(P) = \max(r, P) : r \in \mathcal{R}\}$ is a VC class of functions with a square-integrable envelope $\mathbb{E}[\max(\bar{r}, P)^2] < \infty$. Thus, $\mathcal{F}$ is Donsker, so in particular, $\mathbb{G}_n(\cdot)$ is asymptotically equicontinuous, i.e., for every $\varepsilon, \eta > 0$ there is a $\delta > 0$ such that

$$\limsup_{n\to\infty} P\left(\sup_{\rho(f,g)<\delta} |\mathbb{G}_n(f - g)| > \varepsilon\right) < \eta$$

where $\rho(f,g) = \mathbb{E}[(f(P) - g(P))^2]^{1/2}$. Since $\rho(\hat{f}_n, f) \to_p 0$, the above immediately implies $(A) = o_P(1)$. This result can also be shown directly using symmetrization lemma and a suitable maximal inequality. Further, by the mean-value theorem:

$$\begin{aligned}
(B) \quad &= \sqrt{n}\{\underline{\psi}'(\tilde{F})(\hat{F}(\hat{r}) - F(\hat{r})) - \underline{\psi}'(\bar{F})(\hat{F}(r^*) - F(r^*))\} \\
&= \underline{\psi}'(\tilde{F}) \cdot \sqrt{n}(\{\hat{F}(\hat{r}) - F(\hat{r})\} - \{\hat{F}(r^*) - F(r^*)\}) \qquad (B') \\
&+ (\underline{\psi}(\tilde{F}) - \underline{\psi}(\bar{F})) \cdot \sqrt{n}\{\hat{F}(r^*) - F(r^*)\} \qquad (B'')
\end{aligned}$$

for some $\tilde{F}$ between $\hat{F}(\hat{r})$ and $F(\hat{r})$ and $\bar{F}$ between $\hat{F}(r^*)$ and $F(r^*)$. Note that

$$(B') = \mathbb{G}(\hat{g} - g),$$

where $g(P) = \mathbf{1}(P \leqslant r)$ and $\hat{g}(P) = \mathbf{1}(P \leqslant \hat{r})$, so using the same arguments as for (A), it follows that $(B.1) = o_P(1)$. Finally, since each of $\hat{F}(\hat{r})$, $F(\hat{r})$, and $\hat{F}(r^*)$ converges in probability to $F(r^*)$, we have

$$(B'') = o_P(1) \cdot \sqrt{n}(\hat{F}(r^*) - F(r^*)) = o_P(1).$$

It follows that $(I) = o_P(1)$. By the mean-value theorem and Slutsky's theorem,

$$\sqrt{n}(\hat{\underline{\pi}}^* - \underline{\pi}^*) = \sqrt{n}(\hat{\underline{\pi}}(r^*) - \underline{\pi}(r^*)) + \underline{\pi}'(r^*)\sqrt{n}(\hat{r} - r^*) + o_P(1),$$

so the limiting distribution of $\sqrt{n}(\hat{\underline{\pi}}^* - \underline{\pi}^*)$ is Gaussian.

The asymptotic variance takes a very complicated form due to the dependence between the summands in the above display, so we propose a non-parametric bootstrap procedure in Algorithm 1. The validity of this procedure follows from repeating the above arguments conditionally on the data and applying the Lindeberg-Feller's CLT.

C Tables

C.1 Additional Results

Price Range	[\$100K, \$1.0M]			[\$1.0M, \$10.0M]		
Number of Bidders	$N \in$[2, 10]	$N \in$[6, 40]	$N \in$[2, 40]	$N \in$[2, 10]	$N \in$[6, 40]	$N \in$[2, 40]
$v_0 = 5/6$						
Avg High Estimate	\$541,688	\$316,470	\$434,965	\$3,811,117	\$2,741,925	\$3,291,057
Avg Realized Profit	0.545	1.502	0.998	0.228	0.937	0.573
Minimax regret reserve						
with $\underline{\rho} = 0.5, \overline{\rho} = 0.6$	1.32	1.55	1.42	1.37	1.27	1.32
– Bounds on profit	[0.592, 0.715]	[1.502, 1.548]	[1.039, 1.148]	[0.227, 0.390]	[0.955, 0.977]	[0.603, 0.716]
– 95% CI	[0.458, 0.844]	[1.303, 1.746]	[0.914, 1.269]	[0.142, 0.473]	[0.883, 1.058]	[0.553, 0.767]
Minimax regret reserve						
with $\underline{\rho} = 0.6, \overline{\rho} = 0.7$	1.26	1.44	1.35	1.33	1.20	1.27
– Bounds on profit	[0.584, 0.691]	[1.493, 1.532]	[1.036, 1.128]	[0.224, 0.379]	[0.950, 0.968]	[0.595, 0.696]
– 95% CI	[0.455, 0.816]	[1.287, 1.724]	[0.919, 1.250]	[0.141, 0.458]	[0.871, 1.051]	[0.544, 0.750]
Minimax regret reserve						
with $\underline{\rho} = 0.7, \overline{\rho} = 0.8$	1.21	1.36	1.25	1.32	1.05	1.20
– Bounds on profit	[0.580, 0.673]	[1.508, 1.536]	[1.035, 1.102]	[0.224, 0.374]	[0.951, 0.958]	[0.595, 0.675]
– 95% CI	[0.442, 0.793]	[1.313, 1.728]	[0.910, 1.222]	[0.140, 0.465]	[0.867, 1.039]	[0.542, 0.734]
$v_0 = 2/3$						
Avg High Estimate	\$541,688	\$316,470	\$434,965	\$3,811,117	\$2,741,925	\$3,291,057
Avg Realized Profit	0.711	1.669	1.165	0.394	1.104	0.739
Minimax regret reserve						
with $\underline{\rho} = 0.5, \overline{\rho} = 0.6$	1.26	1.51	1.36	1.15	1.05	1.24
– Bounds on profit	[0.699, 0.834]	[1.644, 1.691]	[1.168, 1.282]	[0.338, 0.474]	[1.114, 1.119]	[0.718, 0.835]
– 95% CI	[0.557, 0.968]	[1.439, 1.894]	[1.041, 1.400]	[0.269, 0.539]	[1.035, 1.202]	[0.667, 0.891]
Minimax regret reserve						
with $\underline{\rho} = 0.6, \overline{\rho} = 0.7$	1.21	1.40	1.27	1.14	1.01	1.18
– Bounds on profit	[0.692, 0.814]	[1.648, 1.685]	[1.159, 1.255]	[0.335, 0.468]	[1.111, 1.115]	[0.715, 0.816]
– 95% CI	[0.546, 0.941]	[1.443, 1.874]	[1.025, 1.364]	[0.267, 0.535]	[1.025, 1.201]	[0.658, 0.876]
Minimax regret reserve						
with $\underline{\rho} = 0.7, \overline{\rho} = 0.8$	1.16	1.32	1.18	1.12	0.85	1.10
– Bounds on profit	[0.694, 0.801]	[1.661, 1.690]	[1.164, 1.237]	[0.336, 0.462]	[1.107, 1.109]	[0.720, 0.798]
– 95% CI	[0.546, 0.947]	[1.467, 1.891]	[1.036, 1.361]	[0.267, 0.538]	[1.021, 1.193]	[0.664, 0.858]
Observations	151	136	287	340	322	662

Table 5: Bounds on expected profit for Modern Art sold in New York City. Figures are scaled by the high estimate, and v_0 is set at 5/6 and 2/3.

C.2 Data Sources

Auction Title	URL
Impressionist and Modern Art Evening Sale — New York	link
20th/21st Century Art Evening Sales — Hong Kong	link
20th Century Evening Sale — New York	link
The Collection of Thomas and Doris Ammann Evening Sale — New York	link
The Collection of Anne H. Bass and 20th Century Evening Sale — New York	link
20th/21st Century: London Evening Sale followed by The Art of the Surreal Evening Sale	link
21st Century Evening Sale — New York	link
A Century of Art: The Gerald Fineberg Collection Part I — New York	link
Post-Millennium Evening Sale — Hong Kong	link
20th/21st Century Art Evening Sales — London	link
Rare Watches Including the Property of Michael Schumacher — Geneva	link
Magnificent Jewels — Geneva	link
20th/21st Century Evening Sales — Hong Kong	link
20th/21st Century: London to Paris — Christies	link
21st Century Evening Sale — New York	link
20th/21st Century Art Auctions — Christie's Hong Kong	link
20th/21st Century: Evening Sale Including Thinking Italian, London — Christies	link
The Cox Collection and 20th Century Evening Sale — New York	link
20th/21st Century: Shanghai to London	link
The Collection of Thomas and Doris Ammann Evening Sale — New York	link
20th/21st Century Art Evening Sales — Hong Kong	link
20th/21st Century: London to Paris Evening Sales	link
20th/21st Century: London	link
The Ann & Gordon Getty Evening Sale — New York	link

Table 6: Christie's YouTube Data Sources

Auction Title	URL
Hong Kong Contemporary Art Evening Sale (LIVE)	link
LIVE from New York — Modern Evening Auction	link
LIVE from New York — The Now and Contemporary Evening Auctions	link
LIVE from Hong Kong — The Now and Modern & Contemporary Evening Auctions	link
LIVE from London — Modern & Contemporary Evening Auction featuring The Now	link
LIVE from New York — The Now and Contemporary Evening Auctions	link
LIVE from New York — The Modern Evening Auction	link
LIVE from New York — The Emily Fisher Landau Collection: An Era Defined Evening Auction	link
LIVE from London — The Now & Contemporary Evening Auctions	link
LIVE from Hong Kong — The Autumn Sales	link
LIVE from London — Freddie Mercury: A World of His Own Evening Sale	link
LIVE from London — Old Master & 19th Century Paintings Evening Auction	link
LIVE — The Now & Modern and Contemporary Auctions, ft. Face to Face: A Celebration of Portraiture	link
The Mo Ostin Collection Evening Auction & The Modern Evening Auction	link
LIVE from New York — The Now & Contemporary Evening Auctions	link
LIVE from London — The Now and Modern & Contemporary Evening Auctions	link
LIVE from New York — The Masters Week Auctions	link
LIVE from New York — Master Paintings & Sculpture Part I	link
LIVE from New York — The David M. Solinger Collection & Modern Evening Auctions	link
LIVE from Paris — Modernits	link
LIVE from London — The Now & Contemporary Evening Auctions	link
LIVE from Paris — Htel Lambert, The Illustrious Collection, Volume I: Chefs-d'oeuvre	link
LIVE from Hong Kong — Modern, Williamson Pink Star & Contemporary Auctions	link
LIVE from London — Old Masters Evening Auction	link
LIVE from London — The Jubilee Auction and Modern & Contemporary Evening Auction	link
LIVE from New York — The Now & Contemporary Evening Auctions	link
LIVE from New York — Modern Evening Auction	link
LIVE from New York — The Macklowe Collection	link
LIVE from New York — Important Watches	link
LIVE from London — Old Masters Evening Sale	link
LIVE From New York — PROUV x BASQUIAT: The Collection of Peter M. Brant and Stephanie Seymour	link
LIVE from New York — Magnificent Jewels	link
LIVE from London — Treasures	link
LIVE from Monaco — KARL, Karl Lagerfelds Estate Part I	link

LIVE from Edinburgh — The Distillers One of One Whisky Auction	link
LIVE from Paris — Art Contemporain Evening Sale	link
LIVE from Sotheby's New York — The Now & Contemporary Evening Auctions With U.S. Constitution Sale	link
LIVE from Sotheby's New York — Modern Evening Auction	link
LIVE from Sotheby's New York — The Macklowe Collection	link
LIVE from Paris — Past/Forward and Modernits	link
LIVE from Las Vegas: Icons of Excellence & Haute Luxury	link
LIVE from Las Vegas — Picasso: Masterworks from the MGM Resorts Fine Art Collection	link
LIVE from New York — Collector, Dealer, Connoisseur: The Vision of Richard L. Feigen	link
LIVE From Sothebys London — Richter, Banksy and Twombley lead the Contemporary Art Evening Auction	link
LIVE From Sothebys Hong Kong — Modern and Contemporary Art Evening Sales	link
LIVE from London — Old Masters Evening Sale	link
LIVE from London: British Art + Modern & Contemporary Auctions	link
LIVE from Hong Kong: Jay Chou x Sothebys — Evening Sale	link
LIVE From Sothebys New York — Important Watches	link
LIVE from Sothebys New York — Magnificent Jewels	link
LIVE from Sothebys Paris — Important Design: from Noguchi to Lalanne	link
LIVE from Sothebys New York — Monet, Warhol and Basquiat Lead Marquee Evening Sales	link
LIVE From Sothebys Hong Kong — Contemporary Art Evening Sale	link
LIVE From Sothebys Hong Kong — Icons and Beyond Legends: Modern Art Evening Sale	link
LIVE from Sotheby's Impressionist & Modern Art + Modern Renaissance Auctions	link
LIVE from Sotheby's Sales of Important Chinese Art and Chinese Art from the Brooklyn Museum	link
LIVE from Sotheby's: The Collection of Hester Diamond Auction in New York	link
LIVE from Sotheby's Master Paintings & Sculpture Auction in New York	link
LIVE from Sotheby's London Old Masters Evening Sale	link
LIVE from Sotheby's marquee Evening Sales of Contemporary and Impressionist & Modern Art	link

Table 7: Sotheby's YouTube Data Sources

C.3 Buyer's Premiums

Saleroom Location	Christie's		Sotheby's	
	Threshold	Rate	Threshold	Rate
Hong Kong	≤HK$7.5M	26.0%	≤HK$7,500,000	26.0%
	>HK$7.5M and ≤ HK$50M	20.0%	>HK$7.5M and ≤HK$40M	20.0%
	>HK$50M	14.5%	>HK$40M	13.9%
London	≤700k	26.0%	≤800k	26.0%
	>700,000 and ≤4.5M	20.0%	>800k and ≤3.8M	20.0%
	>4.5M	14.5%	>3.8M	13.9%
Paris	≤700k	26.0%	≤800k	26.0%
	>700,000 and ≤4M	20.0%	>800k and ≤3.5M	20.0%
	>4M	14.5%	>3.5M	13.9%
New York	≤$1M	26.0%	≤$1M	26.0%
	>$1M and ≤$6M	20.0%	>$1M and ≤$4.5M	20.0%
	>$6M	14.5%	>$4.5M	13.9%
Shanghai	≤6M	26.0%	-	-
	>6M and ≤40M	20.0%	-	-
	>40M	14.5%	-	-

This table is accurate as of February 7 2022 for Christie's and February 1 2023 for Sotheby's. In the last 10 years, there are only minor changes to the base rate (i.e. lowest threshold category). These buyer premium thresholds are additive, so final transaction amounts are strictly increasing. *Source:* Christie's and Sotheby's Websites.

Table 8: Buyer's Premiums in Christie's and Sotheby's Auctions